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Non-Cooperative Games for Outsourcing Operations

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Abstract

Consider a set of manufacturers all of which can outsource part of their workload to a third party. For simplicity, we assume that every manufacturer as well as the third party each possess a single machine. Each manufacturer has to decide the amount of workload to be outsourced so as to minimize the completion time of his in-house and outsourced workloads. In an effort to provide good service to all, the third-party (P) gives priority to manufacturers whose outsourced workload is small. This incentive scheme forces manufacturers to compete for position in the third-party processing sequence. In the first part of the article we develop pure Nash equilibria schedules under 3 distinct protocols for production and 4 for information sharing among manufacturers. Subsequently, we study the impact of information on player strategies. We find that information sharing is beneficial to all in the production chain and that near optimal performance can be achieved without the burden of centralized control if P chooses an appropriate incentive rule and the manufacturers share sufficient amounts of information about their job profiles.

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1 Introduction

Industrial and service organizations increasingly use third parties to outsource part of their operations so that they can deliver goods and services faster to their customers. As a result a third party typically serves multiple manufacturers or service providers which in return forces manufacturers to compete for the use of the third party capacity so as to serve their individual objective. The third party on the other hand has her own objective which may be related to overall service to the manufacturers, or to the utilization of her manufacturing/service capacity. Conflicting interests amongst manufacturers and the third party endow the opportunity of outsourcing with a serious capacity allocation problem.

One can think of 2 ways to resolve the capacity management problem created by outsourcing to a third party: cooperation or competition. The literature on cooperation of operations suggests that a centralized minimum cost solution be used together with incentive payment schemes that make the centralized schedule more profitable than any of the alternatives for every member of the production chain. The concept of competition suggests that the third party announces a priority rule to the manufacturers who in turn decide independently the amount of work that should be outsourced so as to maximize their individual benefit.

Cooperation and competition themselves pose new challenges. For cooperation the challenge is to process and analyze large amounts of information collected from many sources in different locations, solve a large problem involving all manufacturers, and find a "fair" incentive payment scheme that will convince all parties to coalesce. Evidently, this is a tall order that only recently has been made possible due to Internet technologies that make the exchange of information fast and inexpensive. Competition avoids many of these challenges but requires the third party to devise a priority rule that will serve her own objective and at the same time render her capacity beneficial to all manufacturers.

Concepts of cooperation and competition have received significant attention since the 1950's, originally in the context of economics in industrial organization and subsequently in operations management. Since the 1980's, these concepts are responsible for a large body of literature on supply chain management (see e.g. Cachon, 2003) where the primary focus is on coordination of inventories. Concepts of competition in relation to inventory management also have been heavily studied with emphasis on the cost of the (decentralized) equilibrium solution as compared to the centralized optimal; see e.g. Lariviere and Porteus (2001) and Cachon (2003).

Research at the shop floor level is conspicuously scarce even though supply coordination necessitates the coordination of production activities. Cooperative scheduling games were introduced by Curiel *et al.* (1989) who considered a single machine game with weighted completion time objective for each player. They showed that the corresponding sequencing game is convex which implies that a reasonable payment scheme called *the core* of the game is guaranteed to exist; see Shapley (1971). Curiel *et al.* (2002) present a survey of sequencing games and consider core issues. A typical model in this literature assumes a single machine and one job per player. Therefore, the results produced are primarily of theoretical interest. Aydinliyim and Vairaktarakis (2006) considered a model motivated by coordination of manufacturing operations at Cisco's supply chain. A set of manufacturers outsource operations to a third-party who books her capacity at a cost. Knowing these costs, manufacturers book available production days in a first-come-first-served order so as to minimize booking plus weighted flow-time costs. Subsequently, the third party creates savings by coordinating operations and devises an allocation scheme so that the coordinated schedule is more profitable to all chain members.

We are not aware of any article dealing with competition at the shop floor production level except possibly the work of Hain and Mitra (2004) where each manufacturer outsources a single job to a third party who is committed to process jobs in nondecreasing order of processing times (or SPT order). In an effort to gain processing priority, each manufacturer has the incentive to quote smaller than actual processing time for his job, the validity of which cannot be verified by the third party. To resolve this problem, the authors develop a money transfer mechanism based on the job durations announced by the manufacturers. The mechanism is such that every chain member is better off announcing his true processing requirement thus ensuring the third party of the SPT order on her facility.

In our models, players outsource part of their workload to improve their overall completion time or *makespan*. As a result, outsourcing is not viewed as a penalty but as a reward. When outsourced jobs incur a penalty, the problem may be viewed as one of *scheduling with rejection*. In such models a job may be rejected from scheduling on a production system and incur a job-dependent penalty. This notion was introduced in Bartal *et al.* (1996) by studying the problem of minimizing the makespan plus the total cost of rejected jobs on multiple processors. The authors developed a 2.618-competitive algorithm for the on-line version of the problem. Subsequently, Seiden (2000) developed a 2.388-competitive algorithm when preemption is allowed for jobs scheduled on multiple processors. Sengupta (1999) developed approximation schemes for lateness or tardiness criterion plus the rejection penalties. Engels *et al.* (2003) studied the objective of minimizing total weighted flowtime in addition to rejection penalties. For the single machine problem they developed a fully polynomial-time approximation scheme (FPTAS). For unrelated and uniform machines they adapted existing algorithms to obtain approximation algorithms with tight worst-case error bounds. Finally, Engels *et al.* (2003) developed a technique that yields approximation schemes for the single machine problem subject to precedence constraints and due-dates for the jobs. In a closely related article by Hoogeveen *et al.* (2003), the authors study the problem of scheduling with rejection when preemption is allowed for the jobs. They provide a complete characterization of complexity and approximability for uniform and unrelated machines and for open shops.

In our models, players compete for scheduling priority at the third party machine. There is a large body of literature in priority scheduling of queues where multiple customer types compete for service. In this setting, competition via incentives is considered by Mendelson and Whang (1990) by deriving a closed-form pricing mechanism for various customer types (corresponding to our players), that is incentive-compatible. Namely, the customer arrival rates and the service priorities jointly maximize the expected net present value of the system while being determined on a decentralized basis, by the players. Priority pricing in queues was introduced by Kleinrock (1967) where the trade-off between the delay cost and the price paid by a player is chosen so as to optimize a system-wide objective. In a related article, Edelson and Hildebrand (1975) introduced a no-balking model in which players decide to join the system without observing its current state. They show that, in this case, revenue maximization by individual players coincides with social optimization. Devising pricing schemes that induce individual customers to implement system-optimal solutions is a popular theme in this literature (e.g., Bell and Stidham 1983, Dewan and Mendelson 1990, Lederer and Li, 1994).

Another body of literature related to our models is when players are represented by retailers competing for the scarce resource of a supplier (which represents our third party). Each retailer privately observes his demand and orders stock to the supplier who decides her capacity level. A one-period model with identical retailers is studied in Lee *et al.* (1997) who found that allocating the supplier capacity in proportion to the retailer orders may lead retailers to inflate their orders. In a related article Cachon and Lariviere (1999) study the effect of publicly known allocation mechanisms to the members of the chain. They show that a broad class of mechanisms is prone to order inflation causing the supplier to choose a higher level of capacity compared to a truth-inducing mechanism; yet, much less than the prevailing distribution of orders. They show that there does not exist a truth-inducing mechanism that maximizes total retailer profits, while in many cases true orders can lower profits for the supplier, the chain, and even the retailers.

In the current article we apply the Nash equilibrium concept (see Nash, 1951) to study competition in the context of outsourcing operations. A schedule is said to be a *pure Nash equilibrium* or (for brevity) a *Nash schedule* when the strategy of each player is optimal given the strategies of the other players. In the next section we describe our model as well as the objectives of the third party and the players.

2 The Third-Party Objective

Consider set M of manufacturers (also referred to as *players*) each having known total workload $P_i : i \in M$. Player i has to determine the amount $x_i \leq P_i$ of work to be outsourced to machine F owned by the third party P. The remaining amount $P_i - x_i$ will be processed in-house on machine M_i owned by player $i \in M$. The amount $x_i : \ i \in M$ is referred to as the *strategy* of player i. Player i's objective is to minimize his maximum completion time of the in-house and outsourced portions of his workload, henceforth referred to as the *makespan* of player i. Evidently, the players want to compete for resource F which forces P to announce the rules of engagement.

Let C^i denote the makespan of player $i \in M$ in schedule S and let x_i be the amount of work outsourced by manufacturer $i \in M$ to third party P. Possible objectives for the third party include optimizing the overall service to manufacturers expressed as $\sum_{i \in M} C^i$. Such would be the case for example if the third party is a flexible manufacturing center of a division of the firm serving the needs of various departments. Alternatively, the third party may look out for her own interest and seek to maximize the total workload outsourced by manufacturers in M, i.e., $\sum_{i \in M} x_i$.

Consider an arbitrary order $[1], [2], \ldots, [|M|]$ of manufacturers. If P announces that work due to player [1] will be processed first, then [1] will contribute as much of his workload as possible without regard to other manufacturers. Similar will be the strategy of the next few manufacturers who will occupy all early processing capacity of F. Consequently, subsequent manufacturers cannot benefit from F and will not outsource any of their workload to F. In this case the total workload outsourced to P is expected to be smaller than the alternative, and $\sum_{i \in M} C^i$ is expected to suffer. Therefore, P would like to announce priority rules to manufacturers so that they compete for her capacity more productively.

We study 3 different production protocols: overlapping, preemption and non-preemption (see Pinedo, 2002). Overlapping allows processing parts of a job of player i simultaneously on M_i and F. Preemption allows processing part of a job of player i on M_i and the rest on F, however, not simultaneously. Non-preemption stipulates that preemption is not possible for any job. As we show in the following 2 sections, when overlapping or preemption is allowed equilibrium schedules exist where all manufacturers attain their makespan on M_i, rather than on F. Then,

$$\sum_{i\in M} C^i = \sum_{i\in M}(P_i - x_i)$$

and hence minimizing the service objective $\sum_{i\in M} C^i$ is consistent with maximizing the total workload $\sum_{i\in M} x_i$ outsourced to F (this property does not necessarily hold for non-preemptive schedules). Therefore, with appropriate incentives to manufacturers the third party P can achieve both superior service as well as maximum utilization of her capacity. Throughout this article we assume that P's objective is to maximize the amount $\sum_{i\in M} x_i$ outsourced. However, as we pointed out above, this objective is congruent with minimizing $\sum_{i\in M} C^i$.

Player strategies depend on the information and production protocols (explained next) which in turn affect the rule making process of P. We consider 4 different information protocols:

IP1) Value $|M|$ is disclosed to all manufacturers.

IP2) Values $P_i \ : \ i \in M$ are disclosed to all manufacturers.

IP3) Values P_i and $p^i_{\max} \ : \ i \in M$ are disclosed to all manufacturers.

IP4) Job processing profiles $\{p_{ij} \ : \ j \in N_i\}$, $i \in M$ are disclosed to all manufacturers,

where N_i denotes the job set of player $i \in M$, $p^i_{\max}$ is the processing time of a longest job in N_i, and p_{ij} is the processing time of job $j \in N_i$.

Evidently, IP4 corresponds to complete information. When overlapping is allowed, IP2 is equivalent to complete information because jobs can be preempted and/or processed simultaneously on F and M_i ($i \in M$) irrespective of their exact processing requirements. When preemption is allowed but overlapping is not, then IP3 is equivalent to complete information because $p^i_{\max}$ is a lower bound on the makespan for player $i \in M$ and as will be shown shortly, the remaining workload $P_i - p^i_{\max}$ can always be processed without overlapping. Hence, detailed job processing information does not provide additional preemption opportunities. On the other hand, detailed job information is useful in non-preemptive schedules.

Information sharing is a central issue in the supply chain literature. Typically, in a 2-stage supply chain with multiple retailers (or players) and a single manufacturer (or third party) there are "direct" effects for sharing point-of-sale data between each retailer and the manufacturer, and "indirect" effects due to retailers changing their strategies in reaction to the information revealed by other retailers. The direct effect is a result of *vertical* information sharing (between each retailer and the manufacturer) and the indirect effect a result of *horizontal* information sharing.

Literature on vertical information sharing is significant and includes Bourland *et al.* (1996), Chen (1998), Gavirneni *et al.* (1999), Lee *et al.* (2000), Aviv and Federgruen (1998) and Cachon and Fisher (2000). In our setting, vertical information sharing is inconsequential because the third party schedules players according to a publicly known dispatching rule.

In contrast, the literature on horizontal information sharing is scarce. Padmanabhan and Png (1997) considered the impact of retail competition or demand uncertainty on a manufacturer's decision to accept returns. Li (2002) studied incentives to share information vertically (between each retailer and a manufacturer) in the presence of horizontal competition amongst retailers. Horizontal information sharing is a core issue of the games studied in this article and affects the makespan of all the players as well as the utilization of the third party.

In the next 3 sections we consider competition under complete information for the 3 production protocols respectively. In Section 6 we study the player strategies under incomplete information. The insights drawn from our findings are summarized in Section 7. We conclude in Section 8.

3 Overlapping Allowed

Throughout this section we use x_i^O to denote the equilibrium strategy of player $i \in M$ when overlapping is allowed. As discussed in the previous section the third-party would like to process the workload of manufacturers in M in a predetermined order of P_i's but such an announcement would result to a bipartition of the manufacturers: those that outsource large amounts, and those that are essentially precluded from using F, resulting to inferior overall service as measured by the sum of all makespans. In this section we assume the following:

- Overlapping is allowed for jobs,
- Values $P_i : i \in M$ are disclosed to all manufacturers, and
- Sequencing on F will be done according to the *Incentive Rule* when *Overlap* is allowed (abbreviated as IRO) defined next.

Definition 1 *:* **Incentive Rule IRO:** *If $x_i^O \leq x_k^O$ then player i precedes k on F (i.e., in the shortest processing time order of outsourced workloads); break ties with smaller P_i, then arbitrarily.*

For all players who outsource the same amount x and have the same total workload P_i, in the above definition we choose an arbitrary priority sequence on F in order to maintain transitivity amongst the players. Incentive IRO gives priority to a player who outsources to P a small amount of workload. This rule presents each manufacturer i with a dilemma. If he outsources a small amount x_i^O, his remaining workload $P_i - x_i^O$ is likely large and his makespan will probably be attained on M_i. If x_i^O is large, then it is likely to be preceded by several other manufacturers, thus delaying the completion of his workload on F and missing on the opportunity to effectively utilize the capacity of F. Hence, the amount x_i^O contributed by each manufacturer provides a framework for competition.

Lemma 1 *There exists a Nash schedule where $x_i^O \leq \frac{P_i}{2}$ for $i \in M$.*

Proof: Consider a Nash schedule S where $x_i^O > \frac{P_i}{2}$ for some $i \in M$. Then, the workload of player i on M_i is $P_i - x_i^O$ and his makespan C^i is attained on F. Hence, $C^i \geq x_i^O$. Let y_i be the workload of player i processed during the time interval $[P_i - x_i^O, C^i]$. By definition, $0 < y_i \leq x_i^O$. Also,

$$C^i \geq P_i - x_i^O + y_i$$

because the y_i units start on F after time $P_i - x_i^O$. Then, remove the $y_i/2$ units of workload from F and schedule them on M_i; let $\tilde{S}$ be the resulting schedule and $\tilde{C^i}$ be the new makespan of player i. Clearly,

$$\tilde{C^i} \leq C^i - \frac{y_i}{2} < C^i$$

and hence the makespan of player i in $\tilde{S}$ is better than in S; contradiction to S being a Nash schedule. □

The following result suggests that, while every player looks out only for himself, in any Nash schedule the sequencing of players at the third-party machine is in nondecreasing order of P_i's. Moreover, the later a player appears in the sequence, the greater the amount of workload outsourced to P. The second part of Theorem 1 suggests that the makespan of player i is attained on M_i for every $i \in M$.

Theorem 1 *Let $P_{[1]}, P_{[2]}, \ldots, P_{[|M|]}$ be a nondecreasing order of player workloads. Then, every Nash schedule is such that $x^O_{[1]} \leq x^O_{[2]} \leq \ldots \leq x^O_{[|M|]}$. Moreover,*

$$\sum_{k=1}^{i} x^O_{[k]} \leq P_{[i]} - x^O_{[i]} \quad for \quad every \quad i \in M.$$

Proof: For contradiction, let S be a Nash schedule such that $P_{[i]} \leq P_{[i+1]}$ but $x^O_{[i]} > x^O_{[i+1]}$ for some $1 \leq i < m$. Then, consider the difference $\delta = (x^O_{[i]} - x^O_{[i+1]})/2$. In S, player $[i+1]$ is processed before $[i]$ on F because $x^O_{[i+1]} < x^O_{[i]}$. Let $t,\ t'$ be the completion times for player $[i+1]$ on machines $M_{[i+1]},\ F$ respectively. If $t' < t$ player $[i+1]$ could increase his workload on F by

$$\delta' = \min\{\frac{|t-t'|}{2}, \delta\}.$$

This would reduce his makespan by $\delta'/2$ and wouldn't change his processing order on F because $x^O_{[i+1]} + \delta' \leq x^O_{[i+1]} + \delta < x^O_{[i]}$ while obviously $x^O_{[i-1]} \leq x^O_{[i+1]}$ before the reallocation and hence $x^O_{[i-1]} \leq x^O_{[i+1]} + \delta'$ afterwards.

On the other hand, if $t' > t$ player $[i+1]$ could decrease his workload on F by δ' and improve his makespan by $\delta'/2$ and possibly have his workload scheduled earlier on F because $x^O_{[i+1]} - \delta' < x^O_{[i+1]}$. In either case player $[i+1]$ could improve his objective contradicting that S is a Nash schedule. Therefore, it must hold that $t' = t$ and the Gantt diagram for $M_{[i]},\ M_{[i+1]},\ F$ has the configuration depicted in Figure 1. Note that $P_{[i]} \leq P_{[i+1]}$, $x^O_{[i+1]} < x^O_{[i]}$ implies $P_{[i]} - x^O_{[i]} < P_{[i+1]} - x^O_{[i+1]}$. But then, player $[i]$ can improve his makespan by reducing

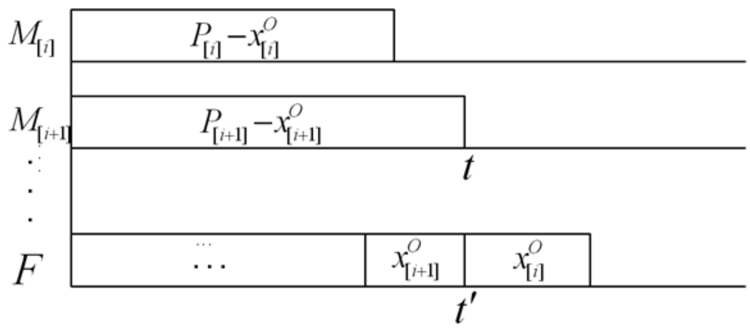

Figure 1: Schedule configuration for players $[i],\ [i+1]$.

his workload $x^O_{[i]}$ on F; contradiction to the fact that S is Nash. This completes the first part of the theorem.

An argument similar to proving $t' = t$ and Lemma 1 yield that player i attains his makespan on M_i, $i \in M$. Equivalently, $\sum_{k=1}^{i} x^O_{[k]} \leq P_{[i]} - x^O_{[i]}$. $\square$

Theorem 2 *In every pure Nash equilibrium, the player strategies are*

$$x^O_{[k]} = \min_{i \geq k} \frac{P_{[i]} - x^O_{[1]} - \ldots - x^O_{[k-1]}}{i + 2 - k} \quad \textit{for} \quad k = 1, 2, \ldots, |M|.$$

Proof: Given strategies $x_{[1]}, x_{[2]}, \ldots, x_{[|M|]}$ for the players, the makespan $C^{[i]}$ of player $[i]$ at equilibrium is $C^{[i]} = \max\{\sum_{k=1}^{i} x_{[k]}, P_{[i]} - x_{[i]}\} = P_{[i]} - x_{[i]} \quad \forall\ i \in M$ due to Theorem 1. Hence, $x_{[1]} + \ldots + x_{[i-1]} + 2x_{[i]} \leq P_{[i]}$ and $x_{[1]} \leq \ldots \leq x_{[i]}$ due to IRO. Then, $(i+1)x_{[1]} \leq P_{[i]}$ or $x_{[1]} \leq \frac{P_{[i]}}{i+1}$ for $i \geq 1$. From

$$x_{[1]} + x_{[2]} + \ldots + x_{[i]} \quad \leq \quad P_{[i]} - x_{[i]} \tag{1}$$

and $i = 1$ we see that the larger $x_{[1]}$, the smaller the makespan $C^{[1]} = P_{[1]} - x_{[1]}$ for player [1]. Therefore,

$$x^O_{[1]} \quad = \quad \min_{i \geq 1} \frac{P_{[i]}}{i+1}. \tag{2}$$

For $x_{[1]} = x^O_{[1]}$ inequality (1) yields

$$
\begin{aligned}
x^O_{[1]} + x_{[2]} + \ldots + x_{[i]} &\leq P_{[i]} - x_{[i]} \\
x_{[2]} + \ldots + 2x_{[i]} &\leq P_{[i]} - x^O_{[1]}
\end{aligned}
$$

or $ix_{[2]} \leq P_{[i]} - x^O_{[1]}$ for $i \geq 2$ (due to IRO), or $x_{[2]} \leq \frac{P_{[i]} - x^O_{[1]}}{i}$. From (1) and $i = 2$ we observe that the larger $x_{[2]}$ the smaller the makespan $C^{[2]} = P_{[2]} - x_{[2]}$ of player [2]. Therefore,

$$
x^O_{[2]} = \min_{i \geq 2} \frac{P_{[i]} - x^O_{[1]}}{i}. \tag{3}
$$

Iterative arguments and expressions (2), (3) yield the result. □

In the above theorem we do not claim that there exists a unique Nash schedule because of possible ties $P_i = P_j$ for players $i \neq j$. However, whatever nondecreasing P_i order is chosen, the strategies of Theorem 2 yield a unique pure Nash equilibrium for that order.

Example 1: To illustrate the above results consider problem instance with $|M| = 4$ and $P_1 = 12$, $P_2 = 14$, $P_3 = 18$ and $P_4 = 23$. The 4 players will be sequenced on F in the order $1, 2, 3, 4$ and the outsourced workloads are $x^O_1 = \min\{\frac{12}{2}, \frac{14}{3}, \frac{18}{4}, \frac{23}{5}\} = 4.5$. If player 1 did not hedge against other manufacturers and were processed first, he would prefer to outsource $\frac{12}{2}$ units for a resulting makespan $C^1 = 6$. Then, player 2 would outsource $\frac{14+6}{2} - 6 = 4$ units which according to IRO would place him ahead of player 1 thus increasing C^1 to $4 + 6 = 10$. Similar arguments hold for player 1 as he hedges against players 3 and 4. Note that the minimand in the expression for x^O_1 is attained at player 3 (because $\frac{18}{4} = 4.5$) indicating that, unless $x^O_1 = 4.5$, player 3 will be processed ahead of 1 on F resulting to inferior makespan for player 1. Similarly, $x^O_2 = \min\{\frac{14-4.5}{2}, \frac{18-4.5}{3}, \frac{23-4.5}{4}\} = 4.5$ - again player 3 restrains player 2 from outsourcing more than 4.5 units - $x^O_3 = 4.5$ and $x^O_4 = 4.75$. Note that according to IRO, player 4 would not be processed prior to player 3 even if 3 outsourced as much as $\frac{4.5+4.75}{2} = 4.625$. However, given that players 1 and 2 have already outsourced the maximum possible according to IRO, it is not beneficial to player 3 to outsource more than 4.5 because when $x^O_1 = x^O_2 = x^O_3 = 4.5$ makespan $C^3 = 13.5$ is attained on both F and M_3. Finally, $\sum_i x^O_i = 18.25$ and $\sum_i C^i =$

$\max\{7.5, 4.5\} + \max\{9.5, 9\} + \max\{13.5, 13.5\} + \max\{18.25, 18.25\} = 48.75 = \sum_i P_i - \sum_i x_i^O$; the latter equality due to Theorem 1.

4 Preemption Allowed

Let N_i be the set of jobs that belong to manufacturer $i \in M$, $p^i_{\max}$ be the processing time requirement of a longest job in N_i and $P_i = \sum_{j \in N_i} p_{ij}$ be the total workload of manufacturer $i \in M$. When preemption is allowed but overlapping is not, each manufacturer will generally follow a different strategy, say $x_i^P : i \in M$. In this section we assume the following:

- Preemption is allowed for jobs but overlapping is not,
- Values $P_i,\ p^i_{\max} \ : \ i \in M$ are disclosed to all manufacturers,
- Sequencing on F will be done according to the *Incentive Rule* when *Preemption* is allowed (abbreviated as IRP) defined next.

Definition 2 *:* **Incentive Rule IRP:** *The manufacturer workloads x_i^P will be processed in quasi-SPT order, i.e., player i precedes k on F if $x_i^P \leq x_k^P$ (break ties with smaller P_i, then arbitrarily), unless $x_i^P = P_i - p^i_{\max} \leq x_k^P$ and $P_i > P_k$.*

In the above definition we break ties only after all outsourced workloads are reported. All players outsourcing the same amount x are ordered in nondecreasing order of P_i's and in case several x, P_i pairs are identical, we simply choose an arbitrary priority subsequence. This avoids transitivity issues. Incentive IRP coincides with IRO except when the *disposable* workload $P_i - p^i_{\max}$ of player i forces him to outsource to P a small amount. In this case, delaying i according to IRP does not hurt player i (as we will see below), allows other players to outsource more workload than they otherwise would, and provides better overall service by letting P utilize the SPT rule with respect to P_i's rather than the SPT rule with respect to x_i^P's. As for the players, in choosing their strategy they face the same dilemma as in the case when overlapping is allowed.

Lemma 2 *If a pure Nash equilibrium exists, then there exists one where the player strategies satisfy the following:*

(a) $x_i^P \leq \min\{P_i - p_{\max}^i, \frac{P_i}{2}\}$ *and*

(b) $x_{[1]}^P + \ldots + x_{[i]}^P \leq P_{[i]} - x_{[i]}^P$ *for every* $i \in M$.

Proof: We first consider property (a). Let S be a Nash schedule. We first prove $x_i^P \leq \frac{P_i}{2}$ by revising S (if necessary) so that the property holds for every player. Suppose that there exists player i with $x_i^P > \frac{P_i}{2}$ who attains his makespan (say C^i) on F while the completion time on M_i is $t_i = P_i - x_i^P < \frac{P_i}{2} < x_i^P \leq C^i$. Then, we can assign on M_i the workload processed on F during the interval $[t_i, C^i]$. This reallocation does not increase the makespan of player i and the revised outsourced workload, say x_i', is no more than $t_i = P_i - x_i^P < \frac{P_i}{2}$, i.e., $x_i' \leq \frac{P_i}{2}$ in the revised schedule. Hence, there exists a Nash schedule such that $x_i^P \leq \frac{P_i}{2}$ for all $i \in M$ - let S be such a schedule. If $P_i - p_{\max}^i \geq \frac{P_i}{2}$ (i.e., $p_{\max}^i \leq \frac{P_i}{2}$) then $\min\{P_i - p_{\max}^i, \frac{P_i}{2}\} = \frac{P_i}{2}$ and (a) holds trivially. Hence, it suffices to consider the case where $p_{\max}^i > \frac{P_i}{2}$ for some player i.

Let job $j \in N_i$ attain $p_{ij} = p_{\max}^i$. If j is processed exclusively on M_i, we have $x_i^P < P_i - p_{ij} < \frac{P_i}{2}$ and (a) holds. If p_{ij} is not processed on M_i exclusively then the makespan of player i is $C^i \geq p_{ij} > \frac{P_i}{2}$ since no overlapping is allowed in S. Let $y_1,\ y_2$ be the workload portions of p_{ij} processed on $M_i,\ F$ respectively. Then, exchange the y_2 periods of time that F is busy with p_{ij}. This exchange preserves the property that jobs of player i are not processed simultaneously on M_i and F. Also, the makespan of i does not increase because the y_2 time units of p_{ij} are processed at the same time - just on a different machine. Also, the total workload of i processed on F does not increase and (according to IRP) i is not scheduled later on F. However, after the exchange the job j is processed entirely on M_i and hence

$$x_i^P \leq P_i - p_{ij} = P_i - p_{\max}^i = \min\{P_i - p_{\max}^i, \frac{P_i}{2}\}$$

because $p_{\max}^i > \frac{P_i}{2}$. This concludes property (a).

We now consider property (b). For contradiction, suppose S is a Nash schedule that satisfies property (a) and there exists player $[i]$ with

$$x_{[1]}^P + \ldots + x_{[i]}^P \quad > \quad P_{[i]} - x_{[i]}^P. \tag{4}$$

Then, the makespan of player $[i]$ is $C^{[i]} = x^P_{[1]} + \ldots + x^P_{[i]}$. Define $\Delta = \min\{\frac{1}{2}(C^{[i]} - P_{[i]} + x^P_{[i]}), x^P_{[i]}\} > 0$. Replace $x^P_{[i]}$ by $\tilde{x}_{[i]} = x^P_{[i]} - \Delta$. Following the reallocation of his workload, the makespan of player $[i]$ becomes

$$\tilde{C}^{[i]} \leq C^{[i]} - \Delta < C^{[i]}$$

because $\Delta > 0$. If $\Delta = x^P_{[i]}$, then $\tilde{x}_{[i]} = 0$ and no overlapping is possible. Otherwise, one can start processing jobs in $N_{[i]}$ on F, one at a time, starting at time $C^{[i]} - x^P_{[i]}$ continuing until time $\tilde{C}^{[i]}$ possibly preempting a single job say $j' \in N_{[i]}$, continuing with job j' on $M_{[i]}$ at time 0, continuing with the rest of the jobs in $N_{[i]}$ until time $\tilde{C}^{[i]}$, preventing any overlapping due to the fact that $\tilde{x}_{[i]}$ satisfies property (a). Therefore, S can be revised to satisfy property (b) for player $[i]$ or any other player violating property (b). This completes the proof of the lemma. □

Note that there can exist Nash schedules with $x^P_i > \min\{P_i - p^i_{\max}, \frac{P_i}{2}\}$. For example, when N_1 has 2 jobs requiring 1, and 10 time units of processing respectively and all other manufacturers have a single job each requiring 100 units. Then player 1 can outsource 10 units of work (instead of 1) and all other manufacturers would process their job in-house. In this example $x^P_1 > P_1/2$ and $P_1 - x^P_1 < x^P_1$. In contrast, the proof of Lemma 2(b) shows that Nash schedules that satisfy (a) necessarily satisfy (b).

The main finding in this section is that a Nash equilibrium for IRP processes players in SPT order of their P_i values as was the case in Theorem 1. In Theorem 1 this ordering coincides with the SPT order of workloads while in this section we show that it coincides with the quasi-SPT order defined in the beginning of the section. The reason for the quasi-SPT order when overlapping is not allowed is due to the longest job of each manufacturer. In light of Lemma 2, the workload outsourced by player i should not exceed $P_i - p^i_{\max}$. For a very long job $p^i_{\max}$ the amount $P_i - p^i_{\max}$ is small thus commanding an early position in the SPT sequence of manufacturer workloads even though P_i is large. In this case, if P used the SPT order to process manufacturer workloads, player i would be scheduled early, even though machine M_i completes the remaining workload much later, and would unnecessarily delay all subsequent players and force them to outsource less of their workload to P. This action would result in inferior service to all manufacturers (as measured by the sum of all completion times) and would decrease the total amount outsourced to P. Incentive IRP avoids both of these undesirable effects.

Theorem 3 *There exists a Nash schedule where manufacturers are processed on F in nondecreasing order of P_i's, $i \in M$.*

Proof: Let $x_1^P, x_2^P, \ldots, x_{|M|}^P$ be the player strategies in a Nash schedule S chosen so that they satisfy Lemma 2. We prove the theorem by establishing that: a) If $P_i \geq P_k$ and $P_i - p_{\max}^i \geq P_k - p_{\max}^k$, then $x_i^P \geq x_k^P$ and player k's workload precedes i's on F, and b) If $P_i \geq P_k$ and $P_i - p_{\max}^i < P_k - p_{\max}^k$, then player k's workload precedes i's on F. The 2 properties imply the result.

We prove claim a) by contradiction. Suppose S is a Nash schedule such that a) holds but $x_i^P < x_k^P$. Then, according to IRP player i precedes k on F as in Figure 2a (where the makespan of i, k is attained on M_i, M_k respectively due to Lemma 2(b)).

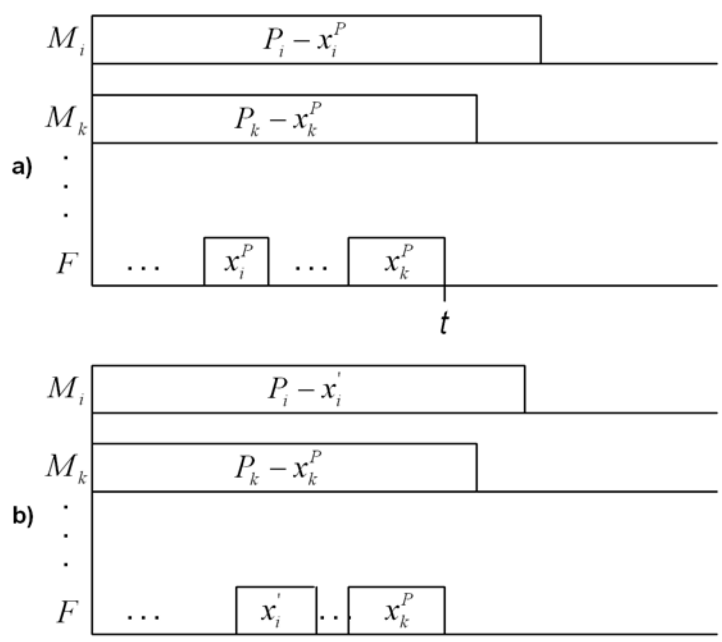

Figure 2: Schedule configurations for property (b).

Since $P_i \geq P_k$ and $P_i - p^i_{\max} \geq P_k - p^k_{\max}$ we have $\min\{\frac{P_i}{2}, P_i - p^i_{\max}\} \geq \min\{\frac{P_k}{2}, P_k - p^k_{\max}\}$ and hence

$$x_i^P < x_k^P \leq \min\{\frac{P_k}{2}, P_k - p^k_{\max}\} \leq \min\{\frac{P_i}{2}, P_i - p^i_{\max}\}.$$

Therefore player i could (if it were beneficial) outsource at least x_k^P units of workload to F. Let $\Delta = \frac{1}{2}(x_k^P - x_i^P) > 0$. Suppose that i reallocates to F $x_i' = x_i^P + \Delta$ units of workload instead of x_i^P. This will result to player i reducing his makespan by Δ even though P may now reschedule player i later - still before player k, as in Figure 2b. In all cases i's makespan decreases by Δ contradicting that S is a Nash schedule. This proves claim a).

To prove claim b) we distinguish 3 cases based on the disposable workload of players i and k as follows.

Case i) $P_i - p^i_{\max} \geq \frac{P_i}{2}$ and $P_k - p^k_{\max} \geq \frac{P_k}{2}$.

In this case the proof goes as for claim a), $x_i^P \geq x_k^P$ and according to IRP player k precedes i on F. Note that in this case $\min\{\frac{P_i}{2}, P_i - p^i_{\max}\} = \frac{P_i}{2}$ and $\min\{\frac{P_k}{2}, P_k - p^k_{\max}\} = \frac{P_k}{2}$.

Case ii) $P_i - p^i_{\max} \geq \frac{P_i}{2}$ and $P_k - p^k_{\max} \leq \frac{P_k}{2}$.

This case is not possible because

$$\frac{P_i}{2} \leq P_i - p^i_{\max} < P_k - p^k_{\max} \leq \frac{P_k}{2}$$

implies $P_i < P_k$; contradiction to assumption $P_i \geq P_k$.

Case iii) $P_i - p^i_{\max} < \frac{P_i}{2}$.

For contradiction, suppose that player i precedes k on F as in Figure 2a. We must assume that $x_i^P < x_k^P$ since otherwise we could schedule the x_i^P units of player i immediately after x_k^P without affecting the makespan of either i or k while ordering the 2 players in nondecreasing order of their total processing. If $x_i^P = P_i - p^i_{\max}$ then player i cannot benefit by outsourcing more workload to P (due to Lemma 2) and (according to IRP) can be rescheduled immediately after player k without affecting anyone's makespan.

If $x_i^P < P_i - p^i_{\max}$ then consider increasing x_i^P to $x_i' = x_i^P + \Delta$ where

$$\Delta = \min\{P_i - p^i_{\max} - x_i^P, \frac{x_k^P - x_i^P}{2}\} > 0.$$

This will improve the makespan of i by Δ but still i will precede player k on F because

$$x'_i = x^P_i + \Delta \le x^P_i + \frac{x^P_k - x^P_i}{2} = \frac{x^P_k + x^P_i}{2} < x^P_k.$$

In all subcases, claim b) holds. This completes the proof of the theorem. $\Box$

Theorem 3 suggests an ordering of manufacturers on F. The following theorem suggests an ordering of their associated workloads.

Theorem 4 *Reindex manufacturers so that*

$$\min\{\frac{P_1}{2}, P_1 - p^1_{\max}\} \le \min\{\frac{P_2}{2}, P_2 - p^2_{\max}\} \le \ldots \le \min\{\frac{P_{|M|}}{2}, P_{|M|} - p^{|M|}_{\max}\}.$$

There exists a Nash equilibrium with $x^P_1 \le x^P_2 \ldots \le x^P_{|M|}$.

Proof: Let x^P_i, $i \in M$ be the player strategies in a pure Nash equilibrium that satisfy Lemma 2 and let S be the associated Nash schedule. We prove the result by contradiction. Suppose that there exist players i, k in S whose workload is processed consecutively on F and are such that

$$\min\{\frac{P_i}{2}, P_i - p^i_{\max}\} \le \min\{\frac{P_k}{2}, P_k - p^k_{\max}\} \quad \textit{but} \quad x^P_i > x^P_k.$$

Let $\Delta = x^P_i - x^P_k$. Consider increasing x^P_k to $x'_k = x^P_k + \Delta$ or decreasing x^P_i to $x'_i = x^P_i - \Delta$. This is doable because

$$x^P_k < x^P_i \le \min\{\frac{P_i}{2}, P_i - p^i_{\max}\} \le \min\{\frac{P_k}{2}, P_k - p^k_{\max}\}.$$

Both reallocations can be made without incurring overlapping as in Lemma 2. Changing either x^P_i or x^P_k will not affect the ordering of i, k on F. If the makespan of either i or k decreases, then S cannot be a Nash schedule. The configuration in Figure 3 is the only configuration where x'_i and x'_k yield worse schedules for both players. This configuration, however, does not correspond to a Nash equilibrium because player k can reduce his workload on F by

$$\delta = \frac{1}{2}\min\{\frac{C^k - P_k + x^P_k}{2}, x^P_k\} > 0$$

and improve his makespan by δ. And since

$$x^P_k - \delta < x^P_k \le \min\{\frac{P_k}{2}, P_k - p^k_{\max}\},$$

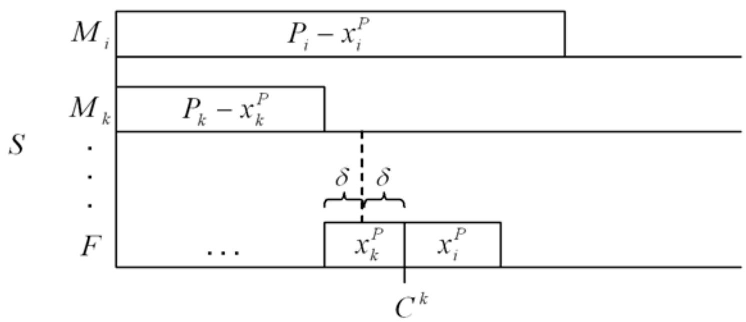

Figure 3: The equilibrium schedule S.

jobs can be rescheduled on M_i so as to avoid overlapping; contradicting the fact that S was Nash before the reallocation. This completes the proof of the theorem. $\Box$

The previous 2 theorems suggest an ordering of manufacturers and workloads on F but not the amount of workload to be contributed by each manufacturer. This is found in the next theorem. But first, we study a key property of pure equilibrium strategies.

Lemma 3 *If $P_k - p^k_{\max} < x^O_k$ then in an equilibrium strategy for player k is $x^P_k = P_k - p^k_{\max}$, otherwise $x^P_k \geq x^O_k$.*

Proof: Theorems 1, 3 indicate that the nondecreasing order of P_i's is the equilibrium order when overlapping or preemption is allowed. This means that the quasi-SPT order stipulated by IRP does not affect the ordering of a player on F when overlaps are not allowed. Consider equilibrium schedules S, S_O with strategies $\{x^P_{[i]} | \ i \in M\}$, $\{x^O_{[i]} | \ i \in M\}$ respectively when overlapping or just preemption are allowed, where $[1], \ldots, [|M|]$ is an SPT order of P_i's. If $x^P_{[k]} \geq x^O_{[k]}$ for every $k \in M$ the result holds trivially. Otherwise, let k be the first index such that $x^P_{[k]} < x^O_{[k]}$. If there exists $i < k$ with $x^P_{[i]} \geq x^P_{[k]}$ then IRP implies that $x^P_{[k]} = P_{[k]} - p^{[k]}_{\max}$ and the result holds. Therefore we assume that $x^P_{[i]} < x^P_{[k]} < x^O_{[k]}$ for all $i < k$. Still, if $x^P_{[k]} = P_{[k]} - p^{[k]}_{\max}$ the result holds. The last 2 observations and the choice of index k imply

$$x^O_{[i]} \leq x^P_{[i]} < x^P_{[k]} < P_{[k]} - p^{[k]}_{\max} \quad for \quad i < k. \tag{5}$$

We now consider the following 2 subcases.

Case i) $\sum_{i=1}^{k} x^P_{[i]} < \sum_{i=1}^{k} x^O_{[i]}$. Then, consider revising the strategy of player $[k]$ to

$$\tilde{x}^P_{[k]} = \min\{x^P_{[k]} + \sum_{i=1}^{k}(x^O_{[i]} - x^P_{[i]}), P_{[k]} - p^{[k]}_{\max}\}.$$

Evidently, $x^P_{[k]} < \tilde{x}^P_{[k]} < x^O_{[k]}$ because $\sum_{i=1}^{k}(x^O_{[i]} - x^P_{[i]}) > 0$ and

$$\tilde{x}^P_{[k]} \leq x^P_{[k]} + \sum_{i=1}^{k}(x^O_{[i]} - x^P_{[i]}) = x^O_{[k]} + \sum_{i=1}^{k-1}(x^O_{[i]} - x^P_{[i]}) < x^O_{[k]}$$

because $x^O_{[i]} < x^P_{[i]}$ for all $i < k$. Moreover, expressions (5) and IRP imply

$$x^P_{[1]} \leq x^P_{[2]} \leq \ldots \leq x^P_{[k-1]} < x^P_{[k]} < \tilde{x}^P_{[k]} < x^O_{[k]}$$

which suggests that the strategies

$$\tilde{x}^P_{[i]} = \min\{x^O_{[i]}, P_{[i]} - p^{[i]}_{\max}\} \quad for \quad i > k$$

are feasible for players $[k+1], \ldots, [|M|]$. Since (according to IRO) $x^O_{[i]} \geq x^O_{[k]}$ for $i > k$, IRP will preserve position $[k]$ for player $[k]$ when using strategy $\tilde{x}^P_{[k]}$.

Consequently, the makespan of player $[k]$ is attained on $M_{[k]}$ because his revised makespan $\tilde{C}_{[k]} = P_{[k]} - \tilde{x}^P_{[k]}$ is such that

$$\begin{aligned}
\sum_{i=1}^{k-1} x^P_{[i]} + \tilde{x}^P_{[k]} &\leq \sum_{i=1}^{k} x^P_i + \sum_{i=1}^{k}(x^O_{[i]} - x^P_{[i]}) = \sum_{i=1}^{k} x^O_{[i]} \\
&\leq P_{[k]} - x^O_{[k]} \qquad due \ \ to \ \ Theorem \ 1 \\
&< P_{[k]} - \tilde{x}^P_{[k]}.
\end{aligned}$$

And since $P_{[k]} - \tilde{x}^P_{[k]} < P_{[k]} - x^P_{[k]}$, the revised strategy $\tilde{x}^P_{[k]}$ is more beneficial to player $[k]$ than strategy $x^P_{[k]}$, which contradicts the assumption that S is a Nash schedule.

Case ii) $\sum_{i=1}^{k} x^P_{[i]} \geq \sum_{i=1}^{k} x^O_{[i]}$. Then, consider strategies $\{x^O_i | \ i \in M\}$ in S_O and consider revising the strategies of players $[1], \ldots, [k]$ to

$$\begin{aligned}
\tilde{x}^O_{[i]} &= \min\{x^P_{[i]}, x^O_{[i]} + \frac{x^O_{[k]} - x^P_{[k]}}{k-1}\} \ \ \forall \ i < k, \ \ and \\
\tilde{x}^O_{[k]} &= x^P_{[k]} < x^O_{[k]}.
\end{aligned}$$

Clearly, $\tilde{x}^O_{[i]} > x^O_{[i]}$ for $i < k$ because by choice of k we have $x^P_{[i]} > x^O_{[i]}$ for $i < k$. Strategies $\tilde{x}^O_{[i]}$ for $i < k$ improve the makespan of players $i < k$ because

$$\sum_{j=1}^{i} \tilde{x}^O_{[j]} \leq \sum_{j=1}^{i} x^P_{[j]} \leq P_{[i]} - x^P_{[i]} < P_{[i]} - x^O_{[i]} \quad for \quad i < k.$$

Also, by definition of $\tilde{x}^O_{[i]}$ for $i \leq k$ and the fact that, in S we have $x^P_{[1]} \leq x^P_{[2]} \leq \ldots < x^P_{[k]}$, we get that

$$\tilde{x}^O_{[1]} \leq \tilde{x}^O_{[2]} \leq \ldots \leq \tilde{x}^O_{[k-1]} \leq \tilde{x}^O_{[k]} = x^P_{[k]} < x^O_{[k]}.$$

The latter observation together with the fact that

$$\sum_{i=1}^{k-1} \tilde{x}^O_{[i]} + x^P_{[k]} \leq \sum_{i=1}^{k-1} (x^O_{[i]} + \frac{x^O_{[k]} - x^P_{[k]}}{k-1}) + x^P_{[k]} = \sum_{i=1}^{k} x^O_{[k]}$$

imply that players $[k+1], \ldots, [|M|]$ can outsource in S_O amounts $x^O_{[k+1]} \leq \ldots \leq x^O_{[|M|]}$ respectively. Since $x^P_{[k]} < x^O_{[k]} \leq x^O_{[i]}$ for $i > k$, players $[1], \ldots, [k]$ maintain their processing priority on F even when using strategies $\tilde{x}^O_{[i]}$ for $i \leq k$. Then the fact that these strategies improve the makespan of players $[1], \ldots, [k-1]$ contradicts that S_O is a Nash schedule.

In both cases we reached a contradiction because $x^P_{[k]} < x^O_{[k]}$. Therefore, in S we must have $x^P_{[k]} \geq \min\{P_{[k]} - p^{[k]}_{\max}, x^O_{[k]}\}$. Equivalently, if $P_{[k]} - p^{[k]}_{\max} < x^O_{[k]}$ then $x^P_{[k]} \geq P_{[k]} - p^{[k]}_{\max}$, i.e., $x^P_{[k]} = P_{[k]} - p^{[k]}_{\max}$ due to Lemma 2(a), otherwise $x^P_{[k]} \geq x^O_{[k]}$. This completes the proof of the lemma. □

Theorem 5 *Let $[1], [2], \ldots, [|M|]$ be a nondecreasing order of P_i's. When preemption is allowed, the strategies $x^P_k = x_k(|M|)$, $k \in M$ obtained by the recurrence relations are in equilibrium*

$$\begin{aligned} \Gamma(0) &= \{k \in M | P_k - p^k_{\max} < x^O_k\} \\ x^P_{[k]}(r) &= \min\{\min_{\substack{k \leq i \leq |M| \\ i \notin \Gamma(r-1)}} \frac{P_{[i]} - x^P_{[1]}(r) - \ldots - x^P_{[k-1]}(r)}{i + 2 - k - n_{k,i}(r-1)}, P_{[k]} - p^{[k]}_{\max}\} \quad for\ k \in M \qquad (6) \\ \Gamma(r) &= \{k \in M | x^P_k(r) = P_k - p^k_{\max}\} \quad for\ r = 1, 2, \ldots, |M|, \end{aligned}$$

where $n_{k,i}(r-1)$ is the number of players amongst $[k], \ldots, [i]$ in $\Gamma(r-1)$.

Proof: Set $\Gamma(0)$ includes players i who (according to Lemma 3) outsource workload equal to $P_i - p^i_{\max}$. This is indeed the case when the values $x^P_{[k]}(1)$ are computed for $k \in M$ because the inside min operator of (6) is null for $i \in \Gamma(0)$. For $k \in M - \Gamma(0)$ workloads

$$x^P_{[k]}(1) = \min_{\substack{k \le i \le |M| \\ i \notin \Gamma(r-1)}} \frac{P_{[i]} - x^P_{[1]}(r) - \ldots - x^P_{[k-1]}(r)}{i + 2 - k - n_{k,i}(r-1)}$$

replicate the optimal strategies in Theorem 2 assuming that the workloads of players in $\Gamma(0)$ are fixed. This is because rule IRO coincides with IRP for players $i \in \Gamma(0)$ who outsource less than $P_i - p^i_{\max}$. Note that, when computing the best strategy $x^P_{[k]}(1)$ for given $k \in M$, the number of players up until player $[i]$ $(k \le i)$ that have not yet outsourced the maximum possible amount (i.e., $P_{[i]} - p^{[i]}_{\max}$) and their workload is not otherwise fixed (i.e., players $[1], [2], \ldots, [k-1]$) are precisely $k-1+n_{k,i}(0)$. Value $n_{k,i}(0)$ is computed using set $\Gamma(0)$. However, Lemma 2(a) suggests that $x^P_{[k]}(1)$ cannot exceed amount $P_{[k]} - p^{[k]}_{\max}$ as enforced in (6).

If none of the players $i \in M - \Gamma(0)$ outsources amount $P_i - p^i_{\max}$ then strategies $x^P_{[k]}(1) | \; k \in M - \Gamma(0)$ are optimal due to Theorem 2 and values $x^P_{[k]}(r)$ remain unchanged for every $k \in M$ and $r = 2, \ldots, |M|$. If on the other hand some player $i \in M - \Gamma(0)$ outsources $P_i - p^i_{\max}$, set $\Gamma(1)$ includes at least one more player and relations (6) revise the optimal strategies of players not in $\Gamma(1)$. After at most $|M|$ iterations, either all players outsource amount $P_i - p^i_{\max}$, or the optimal strategies have been found. In either case, strategies $x^P_{[k]}(|M|)$ are optimal. This completes the proof of the theorem. $\Box$

Example 2: Consider the instance of Example 1 with the additional information that $p^1_{\max} = 5$, $p^2_{\max} = 11$, $p^3_{\max} = 7$ and $p^4_{\max} = 8$. Then, $\Gamma(0) = \{2\}$ because $P_2 - p^2_{\max} = 3 < x^O_2 = 4.5$. Hence, according to Lemma 3 the strategy for player 2 is $x^P_2 = 3$. Knowing $\Gamma(0)$, we obtain $x^P_{[1]}(1) = \min\{\frac{12}{2}, \frac{18-3}{3}, \frac{23-3}{4}\} = 5$ - in the latter 2 fractions $n_{1,3}(0) = n_{1,4}(0) = 1$ is the number of players in $\Gamma(0)$ between players [1] and [3], [4]. Note that the minimand for player [1] is attained by both players [3] and [4] associated with terms $\frac{18-3}{3}, \frac{23-3}{4}$ respectively. This means that if player [1] were to outsource more than 5 units, he would be processed after players [3], [4] thus suffering makespan losses. Moreover, the quantities $\frac{18-3}{3}, \frac{23-3}{4}$ directly account for player [2] who cannot outsource more than 3 units. For player [2] we have $x^P_{[2]}(1) = 3$ because the inside min operator in (6) is null. For the same reason $x^P_{[2]}(r)$ will remain the same for every $1 < r \leq |M|$. Also, $x^P_{[3]}(1) = \min\{\frac{18-5-3}{2}, \frac{23-5-3}{3}\} = 5$ and $x^P_{[4]}(1) = \frac{23-5-3-5}{2} = 5$. In this allocation no new player outsources amount equal to $P_i - p^i_{\max}$, sets $\Gamma(1), \Gamma(2), \Gamma(3)$ equal $\Gamma(0)$, and hence the remaining 3 iterations of the recurrence relation $x^P_i(r)$ won't change the $x^P_i(1)$ values. Therefore, $(x^P_1, x^P_2, x^P_3, x^P_4) = (5, 3, 5, 5)$, $\sum_i x^P_i = 18$ and $\sum_i C^i = \max\{5, 7\} + \max\{8, 11\} + \max\{13, 13\} + \max\{18, 18\} = 49 = \sum_i (P_i - x^P_i)$; the latter due to Lemma 2(b).

Note that player [4] could outsource more than 5 units if this were beneficial, but this is not the case because the cumulative workload outsourced by players [1], [2], [3] is 13 and his total workload is 23. Therefore, 5 units is the most that player [4] could outsource profitably. Also, observe how this solution differs from $(x^O_1, x^O_2, x^O_3, x^O_4) = (4.5, 4.5, 4.5, 4.75)$ obtained in Example 1. Knowing that player [2] cannot outsource more than 3 units, does not let player [1] take all the benefits (i.e., $x^O_2 - x^P_2 = 4.5 - 3 = 1.5$ units) because he still has to hedge his processing order in F. Instead, IRP forces players to distribute the 1.5 units amongst them almost equitably. In particular, [1] and [3] outsource 0.5 units more, while [4] (who in Example 1 outsourced 4.75 units - 0.25 more than [1], [3]) outsources only 0.25 units more. In effect, IRP helped players to equitably utilize F.

Two more observations are in order. First, the recurrence relations in Theorem 5 require $\mathcal{O}(|M|^2)$ effort including updating the $n_{k,i}(r)$ values in each iteration. Second, given a nondecreasing order of P_i's the strategies produced by the recurrence relations of Theorem 5 yield a unique pure Nash equilibrium. Tie-breaks and the implicit requirements of Lemma 2 indicate that alternative Nash equilibria can exist.

5 The Non-Preemptive Problem

In this section we consider Nash equilibria when jobs must be processed without interruption. When preemption is not allowed the player strategies, denoted as x^N_i, are significantly different to those in previous sections. In her effort to motivate better utilization of F and overall service the third party needs an appropriate incentive rule. For this, we need the following operator.

Definition 3

$f_i(w)$ = *the maximum workload of any subset* $A_i \subseteq N_i$ *that does not exceed* w.

Operator $f_i(w)$ is equivalent to a knapsack problem with a knapsack of size w where jobs are the items and job $j \in N_i$ has size p_{ij} and value p_{ij}. In the rest of this section we assume the following:

- Preemption is not allowed for jobs,
- Profiles $\{p_{ij} \ : \ j \in N_i\}_{i \in M}$ are disclosed to all manufacturers,
- Sequencing on F will be done according to the *Incentive Rule* for the *Non-preemptive* problem (abbreviated as IRN) defined next.

Definition 4 *:* **Incentive Rule IRN:** *Manufacturer workloads* x_i^N *will be processed in non-decreasing order of* P_i *subject to the workload constraints* $x_{[i]}^N \geq f_{[i]}(\max_{k<i} x_{[k]}^N), \quad i \in M$*; break ties arbitrarily.*

Breaking ties is done so as to avoid transitivity problems amongst the players who outsource the same amount x by ordering them on F arbitrarily. IRN suggests that, the jobs outsourced by each manufacturer i are those that fit in a knapsack of size $w_i \geq \max_{k<i} x_{[k]}^N$ at least as big as the size of any of his predecessors on F. This rule forces manufacturers with small total workload P_i to outsource to F amounts that do not prevent subsequent players from choosing jobs from a bigger knapsack. Therefore, even though early manufacturers would like to outsource as much as possible, they are forcibly restrained so as to allow subsequent manufacturers the opportunity to outsource even more. And since the workload of later manufacturers is larger, it is likely that $\sum_{i \in M} x_i^N$ is larger thus benefiting P. It is easy to observe that IRN reduces to IRO, IRP in case that overlapping or preemption are allowed respectively. In the latter case, it suffices to note that for every $w \ : \ \min\{P_i - p_{\max}^i, \frac{P_i}{2}\} \leq w \leq \frac{P_i}{2}$, we have $f_i(w) = f_i(\min\{P_i - p_{\max}^i, \frac{P_i}{2}\})$. The equilibrium properties of IRN are different than those for IRO and IRP. Instead, we make the following observations.

Lemma 4 *Let* $x_1^N, x_2^N, \ldots, x_{|M|}^N$ *be the player strategies in a Nash schedule* S*. Then,* $\forall \ i \in M$*,*

(a) $x^N_{[1]} + \ldots + x^N_{[i]} - (P_{[i]} - x^N_{[i]}) < \min_{j\in A_i} p_{ij}$,

(b) $x^N_i \le \min\{f_i(\frac{P_i}{2}), P_i - p^i_{\max}\}$.

Proof: To prove (a), let $A_i \subseteq N_i$ denote the subset of jobs outsourced to P by player i. Then, if the last job of player i processed on F completes after time $P_i - x^N_i + \min_{j\in A_i} p_{ij}$, player i would be better off to reschedule the smallest job in A_i on M_i and revise his strategy to $x'_i = x^N_i - \min_{j\in A_i} p_{ij}$. Such reallocation will not worsen his makespan and it may result to improvement because $x'_i < x^N_i$ and hence i may be scheduled earlier on F. The proof of part (b) is similar to Lemma 2(a). □

Lemma 4(a) states that, if the makespan of player i is attained on F, then the outsourced workload cannot complete $\min_{j\in A_i} p_{ij}$ or more time units after the completion time of M_i. This contrasts Theorem 1 and Lemma 2(b) where overlaps, preemption are allowed respectively and the makespan of every manufacturer is attained on his processor.

In what follows we develop a dynamic programming algorithm to find an equilibrium schedule with respect to IRN. For notational simplicity we assume that the players are ordered in nondecreasing order of P_i's. We define the following state variables.

Definition 5

$f_{ij}(x, l)$: *the maximum outsourced workload amongst the first j jobs of player i (that does not exceed x) when the smallest job outsourced is $l \le j$; $i \in M$ and $x \le \min\{\frac{P_i}{2}, P_i - p^i_{\max}\}$.*

Values $f(\cdot,\cdot)$ will be used later to verify Lemma 4(a). Let p_{ij} be the processing times of jobs $j \in N_i$, $i \in M$. Then,

$$f_{ij}(x,l) = \max \begin{cases} f_{i,j-1}(x,l) & \\ f_{i,j-1}(x - p_{ij}, l) + p_{ij} & \text{if } p_{il} \le p_{ij} \\ \max_{\substack{1\le l' < j \\ p_{il'} > p_{ij}}} \{f_{i,j-1}(x - p_{ij}, l')\} + p_{ij} & \text{if } l = j \end{cases}$$

together with boundary conditions $f_{i1}(x, l) = p_{i1}$ if $x = p_{i1}$ and $l = 1$; $-\infty$ otherwise, produces all possible strategies of player $i \in M$. This is because the first branch of the program reflects the case when job $j \in N_i$ is not outsourced to P, the second when $j \in N_i$ is outsourced but is not smaller than job l, while branch 3 captures the case when $j \in N_i$ is the smallest outsourced job among jobs $1, 2, \ldots, j \in N_i$. Evidently, the effort to compute $f_{ij}(x, l)$ is $\mathcal{O}(|N_i|)$ and the size of the state space is $\mathcal{O}(|N_i|^2 P_i)$ for every $i \in M$. Hence, the total effort to compute the values for all $i \in M$ is bounded by $\mathcal{O}(\max_i |N_i|^3 |M| P_n)$. Clearly, values $f_i(w)$, $f_{ij}(x, l)$ are related by $f_i(w) = \max_{\substack{l \in N_i \\ 0 \le x \le w}} f_{i,|N_i|}(x, l)$. Subsequently, one can define the candidate values for $x_i^N : i \in M$ as follows:

$$X_i = \{0 \le x \le \min\{\frac{P_i}{2}, P_i - p^i_{\max}\} \mid \exists\, l \in N_i \ : \ f_{i,|N_i|}(x, l) = x\}.$$

Knowing sets $X_i \ : \ i \in M$, the following algorithm produces a Nash schedule.

Non-Preemptive Nash (NPN)

Input : Processing time profiles $\{p_{ij} \ : \ j \in N_i\}$ for $i \in M$

Output : Equilibrium workloads $x_i^N \ : \ i \in M$ for the non-preemptive problem

Begin

[1] Determine sets $X_i \ : \ i \in M$ and let $x_i^N := 0$ for $i \in M$

[2] For all $i \in M$ and $x \in X_i$ determine $l_i(x)$ to be a smallest job with $f_{i,|N_i|}(x, l_i(x)) = x$

For $k := 1$ to $|M|$ **do begin**

[3] **For** $i := k$ to $|M|$ **do** compute $F_{ki}(x) = \sum_{j=1}^{k-1} x_j^N + \sum_{j=k}^{i} f_{j,|N_j|}(x, l(x))$

for all $x \in X_k : \ x \ge x_{k-1}^N$

[4] Let $x \in X_k$ be the smallest value that solves $\min_{x' \in X_k} \max\{P_k - x', \sum_{j=1}^{k-1} x_i^N + x'\}$

[5] subject to constraints $F_{ki}(x) < P_i - f_{i,|N_i|}(x, l(x)) + p_{i,l(x)}$ for every $i > k$

Set $x_k^N := x$

End

End

In line [3] we compute the completion time of player i on F when players k through i outsource $x \ge x_{k-1}^N$, i.e., when IRN is satisfied. In line [4] player k identifies his best strategy $x \in X_k$ given $x_1^N, \ldots, x_{k-1}^N$. Line [5] ensures that the choice made in line [4] allows subsequent manufacturers to utilize knapsacks of size at least x and satisfies Lemma 4(a). The smallest $x \in X_k$ that satisfies [4], [5] is the equilibrium strategy for player k.

The effort expended in line [2] for each $i \in M$ is $\mathcal{O}(|N_i||X_i|)$ resulting to $\mathcal{O}(\sum_i |N_i||X_i|)$ for all manufacturers which is bounded by $\mathcal{O}(|M| \max_i |N_i|P_n)$. Every visit of line [3] requires effort $\mathcal{O}(|M||X_k|)$ when the values $f_{j,|N_j|}(x, l(x))$ are stored appropriately. Accounting for the k-loop the total effort expended is bounded by $\mathcal{O}(|M|^2 P_n)$. Lines [4], [5] require $\mathcal{O}(|X_k||M|)$ for every value of k resulting to overall effort $\mathcal{O}(\max_i |X_i||M|^2)$ which is bounded by $\mathcal{O}(|M|^2 P_n)$. In general, it is expected that $|M| < P_n, \max_i |N_i|$ and hence, the overall complexity of NPN is dominated by $\mathcal{O}(\max_i |N_i||M|P_n)$. This effort is in addition to $\mathcal{O}(\max_i |N_i|^3 |M| P_n)$ required for the values $f_{ij}(x, l)$ and hence this is also the complexity of algorithm NPN. The following result validates the correctness of NPN.

Theorem 6 *Algorithm NPN produces a pure equilibrium.*

Proof: We prove that the strategies $x_i^N : \ i \in M$ are in equilibrium by contradiction. Let $i_0 \geq 1$ be the first player who can improve his makespan by revising his strategy $x_{i_0}^N$ to, say, x_{i_0}. Set X_i includes all strategies of player $i \in M$ and hence $x_{i_0}^N, x_{i_0} \in X_{i_0}$. We first consider the case where $x_{i_0} < x_{i_0}^N$. Then, $P_{i_0} - x_{i_0} > P_{i_0} - x_{i_0}^N$ and since x_{i_0} is an improving strategy the makespan of i_0 under strategy $x_{i_0}^N$ is attained on F; not M_{i_0}. Specifically, strategy x_{i_0} decreases the makespan of i_0 on F by $x_i^N - x_{i_0}$, increases i_0's completion time on M_{i_0} by the same amount and since it is improving,

$$\sum_{i=1}^{i_0} x_i^N - (P_{i_0} - x_{i_0}^N) > x_{i_0}^N - x_{i_0} \quad or \quad \sum_{i=1}^{i_0} x_i^N > P_{i_0} - x_{i_0}.$$

Therefore,

$$\max\{P_{i_0} - x_{i_0}, \sum_{i=1}^{i_0-1} x_i^N + x_{i_0}\} < \sum_{i=1}^{i_0} x_i^N = \max\{P_{i_0} - x_{i_0}^N, \sum_{i=1}^{i_0} x_i^N\}$$

contradicting the choice of $x = x_{i_0}^N$ in line [4].

Consider now the case when $x_{i_0} > x_{i_0}^N$ is an improving strategy. Same rationale as before indicates that the makespan of player i_0 must be attained on M_{i_0} when using strategy $x_{i_0}^N$,

$$P_{i_0} - x_{i_0}^N - \sum_{i=1}^{i_0} x_i^N > x_{i_0} - x_{i_0}^N \quad and$$

$$\max\{P_{i_0} - x_{i_0}, \sum_{i=1}^{i_0-1} x_i^N + x_{i_0}\} < P_{i_0} - x_{i_0}^N = \max\{P_{i_0} - x_{i_0}^N, \sum_{i=1}^{i_0} x_i^N\}. \tag{7}$$

In this case the workload of every player $i > i_0$ is delayed by $x_{i_0} - x_{i_0}^N$. Despite (7), strategy x_{i_0} is not chosen in line [4] of NPN. This suggests that at least one of the inequalities in [5] are violated for x_{i_0}. Inequalities in [5] correspond to Lemma 4(a). Suppose that a violated inequality corresponds to player $k_0 : i_0 < k_0 \leq |M|$. Then, $F_{i_0,k_0}(x_{i_0}) \geq P_{k_0} - f_{k_0,|N_{k_0}|}(x_{i_0}, l(x_{i_0})) + p_{k_0,l(x_{i_0})}$ or

$$\sum_{j=1}^{i_0-1} x_j^N + \sum_{j=i_0}^{k_0} f_{j,|N_j|}(x_{i_0}, l(x_{i_0})) \geq P_{k_0} - f_{k_0,|N_{k_0}|}(x_{i_0}, l(x_{i_0})) + p_{k_0,l(x_{i_0})}.$$

But then, player k_0 would respond by allocating job $p_{k_0,l(x_{i_0})}$ on M_{k_0} rather than F without any loss in makespan. His revised strategy $x_{k_0} - p_{k_0,l(x_{i_0})}$ is an element of X_{k_0} (generated by simply removing $p_{k_0,l(x_{i_0})}$ from the set of outsourced jobs that yielded value $f_{k_0,|N_{k_0}|}(x_{i_0}, l(x_{i_0}))$. This means that strategy x_{i_0} violates IRN (in line [5] of NPN) and hence it is not a feasible option for player i_0. This completes the proof of the theorem. □

It is interesting to observe that the Nash equilibrium produced by NPN is unique if the order of the players is determined as in IRN. However, if $P_{[i]} = P_{[i+1]}$, $f_{[i]}(x)$ is not necessarily equal to $f_{[i+1]}(x)$ and hence the choice $x_{[i]}^N$ greatly affects $x_{[k]}^N$ for $k > i$ resulting to different equilibria. Finally, in line [4] of NPN we select the smallest among the possible optimal strategies in X_k because player k is indifferent among his optimal strategies while subsequent players may outsource more.

Example 3: Consider the instance used in Examples 1 and 2 when the processing time profiles are $\{p_{1j}\} = \{5, 4, 2, 1\}$, $\{p_{2j}\} = \{11, 2, 1\}$, $\{p_{3j}\} = \{7, 6, 5\}$ and $\{p_{4j}\} = \{8, 8, 7\}$. The players will again be processed in the order 1, 2, 3, 4 on F, but the workload of each must come from a knapsack no larger than any of his successors. The set X_1 of 2-partitions for player 1 where one part includes at least half of the total workload is $X_1 = \{6, 5, 4, 3, 2, 1, 0\}$ (e.g. he can outsource workload of 6=4+2, or 5 due to the job of length 5, etc.). If player 1 outsourced 6 units of processing, then player 2 would outsource no less than $f_2(6) = 3$ and player 3 would have to pick jobs from a knapsack of size at least 6. Note that $X_3 = \{7, 6, 5, 0\}$ and hence player 3 would have to outsource 6 or more units of workload. But then, his completion time on F would be $6+3+6=15$ while the remaining workload on M_3 would be 11. Then, the makespan of player 3 would be 15. But this scenario is not an equilibrium for player 3 because he can outsource 5 instead of 6 units and attain makespan $\max\{6+3+5, 7+6\} = 14 < 15$. But then, player 1 violates IRN (which stipulates that the equilibrium choice of player 3 utilizes a knapsack of size no less than the size utilized by player 1, i.e., line [5] of NPN). Hence, $x_1^N = 6$ is not the equilibrium choice for player 1. On the other hand, $x_1^N = 5$ yields $x_2^N = 3$ (because $X_2 = \{3, 2, 1, 0\}$), $x_3^N = 5$ and $x_4^N = 7$ (because $X_4 = \{8, 7, 0\}$ and in line [5], $5 + 3 + 5 + 7 < 23 - 7 + 7$). Evidently, $\sum_i x_i^N = 20$ and $\sum_i C^i = \max\{7, 5\} + \max\{11, 8\} + \max\{13, 13\} + \max\{16, 20\} = 51$.

In this example $\sum_i P_i - \sum_i x_i^N = 47 \neq \sum_i C^i$. Observe that, when preemption is not allowed,

$$\sum_i C^i \geq \sum_i P_i - \sum_i x_i^N$$

because for some player(s) the makespan may be attained on F and hence $C^i > P_i - x_i^N$.

6 Strategies under Incomplete Information

In this section we study the player strategies under incomplete information. We need the following definition.

Definition 6

$x_i^{prod}(IP)$ = *equilibrium strategy of player i under production protocol $prod \in \{O, P, N\}$ for overlapping, preemptive and non-preemptive production respectively, and information protocol $IP \in \{IP1, IP2, IP3, IP4\}$ as described in Section 2.*

In what follows we study equilibrium strategies for each player that won't hurt his "worst case" makespan performance where "worst" is interpreted in terms of the (unrevealed portion of the) processing time profile of other players. Unless specified otherwise, in all prod/IP combinations we assume that every player chooses his strategy under the following assumptions: i) every player has complete information of his own job profile, ii) the workload of all other players is infinitely divisible, and iii) the total workload of other players is the same as his own.

In Table 1 we present equilibrium strategies for each manufacturer $i \in M$ and each production and information protocol combination. Rows O,P,N correspond to production protocols and IP columns correspond to information protocols. According to this (unified) notation, $x_i^O(IP2) = x_i^O$, $x_i^P(IP3) = x_i^P$ and $x_i^N(IP4) = x_i^N$. Stategies x_i^O, x_i^P, x_i^N were verified in

Table 1: Equilibrium strategies for outsourcing operations

Prod/IP	IP1	IP2	IP3	IP4
O	$\frac{P_i}{\lvert M\rvert+1}$	x_i^O	-	-
P	$\min\{P_i - p_{\max}^i, \frac{P_i}{\lvert M\rvert+1}\}$	$\min\{P_i - p_{\max}^i, x_i^O\}$	x_i^P	-
N	$f_i(\frac{P_i}{\lvert M\rvert+1})$	$f_{[i]}(x_{[i]}^O):\ i < \lvert M\rvert$	$f_{[i]}(x_{[i]}^P):\ i < \lvert M\rvert$	$x_{[i]}^N$

Theorems 2, 5 and 6 respectively. Combinations marked with "-" in Table 1 indicate situations where additional information does not affect the player strategies. In what follows we verify all strategies not previously discussed and illustrate them using our example problem. The resulting player strategies for the various production/information protocols are provided in Table 2 where we also include the centralized optimal $\sum_i x_i$ for P, for the overlapping, preemptive and non-preemptive problems.

Table 2: Equilibrium strategies and Centralized optima for example problem

	O			P				N				
$x_{[i]}$	IP1	IP2	Centr	IP1	IP2	IP3	Centr	IP1	IP2	IP3	IP4	Centr
$x_{[1]}$	2.4	4.5	6	2.4	4.5	5	6	2	4	5	5	5
$x_{[2]}$	2.8	4.5	4	2.8	3	3	3	2	3	3	3	3
$x_{[3]}$	3.6	4.5	4	3.6	4.5	5	4.5	0	0	5	5	5
$x_{[4]}$	4.6	4.75	4.5	4.6	4.75	5	4.75	0	7	7	7	7
$\sum_i x_i$	13.4	18.25	18.5	13.4	16.75	18	18.25	4	14	20	20	20
$\sum_i C^i$	53.6	48.75	48.5	53.6	50.25	49	48.75	63	53	51	51	51

6.1 Strategies for IP1

To verify the strategies under IP1, observe that, when $|M|$ is the only piece of information known to manufacturers, each will hedge against the worst case scenario in which all other manufacturers have the same total workload and their jobs are infinitely divisible. Then, when overlapping is allowed, manufacturer i will outsource $\frac{P_i}{|M|+1}$ units of work because even if he is scheduled last, his completion time on F will be $|M|\frac{P_i}{|M|+1}$; i.e., the same as in M_i. If he decided to outsource more and every other manufacturer $k \neq i$ outsourced $\frac{P_k}{|M|+1}$, then his makespan would suffer as seen in Theorem 1. Using strategies $\frac{P_i}{|M|+1}$, the sequencing order of the players according to IRO will be in increasing order of P_i and the total outsourced workload will be $\sum_i \frac{P_i}{|M|+1}$. In our example, under IP1 the players would outsource $\frac{P_1}{5}, \frac{P_2}{5}, \frac{P_3}{5}, \frac{P_4}{5} = 2.4, 2.8, 3.6, 4.6$ units of workload respectively and $\sum_i x_i^O(IP1) = 13.4$ as indicated in Table 2 for the O/IP1 combination. Also, $\sum_i C^i = \max\{9.6, 2.4\} + \max\{11.2, 5.2\} + \max\{14.4, 8.8\} + \max\{18.4, 13.4\} = 53.6 = \sum_i P_i - \sum_i x_i^O(IP1)$.

The same argument can be made when preemption is allowed but overlapping is not. The only difference is that in this case i schedules on F no more than $P_i - p^i_{\max}$ units; information that is not available to other players. In this case P will sequence the players on F (according to IRP) in nondecreasing order of workload which may be different than the nondecreasing order of P_i's. However, this will not affect the makespan of the players because they assume that all players have the same total workload and their strategy results to the same makespan even if scheduled last on F. Hence, the strategy of player i is $\min\{\frac{P_i}{|M|+1}, P_i - p^i_{\max}\}$. In our example, the amounts outsourced are $\min\{2.4, 7\}, \min\{2.8, 3\}, \min\{3.6, 13\}, \min\{4.6, 18\}$, $\sum_i x_i^P(IP1) = 13.4 = \sum_i P_i - \sum_i x_i^P(IP1)$ and $\sum_i C^i = 53.6$ as with the O/IP1 combination.

Similarly, when preemption is not allowed, manufacturer i will outsource $f_i(\frac{P_i}{|M|+1})$ units and hence according to IRN players with larger P_i's pick jobs from a knapsack of size no smaller than any of his predecessors. This doesn't affect the makespan of any player because the completion time of player $[i]$ on F will be no greater than $\frac{iP_i}{|M|+1} \leq \frac{|M|}{|M|+1} P_{[i]}$ which is no greater than his completion time on $M_{[i]}$. In our example, $x_1^N(IP1) = f_1(2.4) = 2$, $x_2^N(IP1) = f_2(2.8) = 2$,

$x_3^N(IP1) = f_3(3.6) = 0$ and $x_4^N(IP1) = f_4(4.6) = 0$, $\sum_i x_i^N(IP1) = 4$ (see Table 2) and $\sum_i C^i = \max\{10,2\} + \max\{12,4\} + 18 + 23 = 63 = \sum_i P_i - \sum_i x_i^N(IP1)$.

6.2 Strategies for IP2

The player strategies for combination O/IP2 were verified in Example 1. Consider now the case when preemption is allowed but overlapping is not. As we saw in Section 4, in this case i would outsource no more than $P_{[i]} - p_{\max}^{[i]}$ units and process at least $\frac{P_{[i]}}{2}$ units on $M_{[i]}$ which allows him to avoid overlaps. If overlapping were allowed $x_{[i]}^O \leq \frac{P_{[i]}}{2}$ would be the maximum amount that player i should outsource. Player $[i]$ will never outsource more than that because he assumes that the workload of all other players is infinitely divisible. Knowing his own profile of processing times, however, $[i]$ will choose strategy $\min\{P_{[i]} - p_{\max}^{[i]}, x_{[i]}^O\}$ and will be scheduled on F according to IRP in quasi-SPT order. Due to Lemma 2(b), his makespan will be attained on $M_{[i]}$ and hence his strategy for combination P/IP2 will not affect his ordering on F. In Example 1 we saw that $x_{[1]}^O = x_{[2]}^O = x_{[3]}^O = 4.5$ and $x_{[4]}^O = 4.75$. Therefore, $x_{[1]}^P(IP2) = \min\{7, 4.5\} = 4.5$, $x_{[2]}^P(IP2) = \min\{3, 4.5\} = 3$, $x_{[3]}^P(IP2) = 4.5$ and $x_{[4]}^P(IP2) = 4.75$. Then, $\sum_i x_i^P(IP2) = 16.75$ (see Table 2) and $\sum_i C^i = \max\{4.5, 7.5\} + \max\{7.5, 11\} + \max\{12, 13.5\} + \max\{16.75, 18.25\} = 50.25 = \sum_i P_i - \sum_i x_i^P(IP2)$.

Similar observations yield that $[i]$ will outsource $f_{[i]}(x_{[i]}^O)$ for combination N/IP2 which does not exceed $P_{[i]} - p_{\max}^{[i]}$. In this case player i is scheduled on F according to IRN and players with larger P_i values outsource workloads picked from larger knapsacks due to Theorem 1. Under IP2, every player assumes that all other players' workload is infinitely divisible and hence cannot take advantage of the fact that $f_{[k]}(x_{[k]}^O)$ may be less than $x_{[k]}^O$. Therefore, $x_{[1]}^N(IP2) = f_{[1]}(x_{[1]}^O) = f_{[1]}(4.5) = 4$, $x_{[2]}^N(IP2) = f_{[2]}(x_{[2]}^O) = f_{[2]}(4.5) = 3$ and $x_{[3]}^N(IP2) = f_{[3]}(x_{[3]}^O) = f_{[3]}(4.5) = 0$. Player $[|M|]$ however, can outsource more than $f_{[|M|]}(x_{[|M|]}^O)$ if this is beneficial to him so as to take advantage of Lemma 4(a).

If there are more than 1 players attaining maximum workload, then they do not know who will be processed last on F and will all outsource amount $f_{[i]}(x_{[i]}^O)$. If there is a unique player

$[|M|]$ with maximum total workload, then he knows his outsourcing choices $X_{[|M|]}$ and will choose the one that minimizes his makespan assuming that the total workload of preceding players is $\sum_{i<|M|} x^O_{[i]}$. Hence, he will choose $x^\star \in X_{[|M|]}$ that solves

$$\min_{x \in X_{[|M|]}} \max\{P_{[|M|]} - x, \sum_{j=1}^{|M|-1} x^O_{[j]} + x\}$$

subject to constraints $\sum_{j=1}^{|M|-1} x^O_{[j]} + x < P_{|M|} - f_{|M|,|N_{[|M|]}|}(x, l(x)) + p_{[|M|],l(x)}$. Evidently, this takes $\mathcal{O}(|N_{|M|}|^3 |M| P_{|M|})$ time due to the $f_{|M|,|N_{|M|}|}(x, l(x))$ values. In our example, player $[4]$ is the only player that attains $P_{[4]} = 23$. He will assume that the preceding players will outsource $x^O_{[1]} + x^O_{[2]} + x^O_{[3]} = 13.5$ units of work even though the actual amount is $x^N_{[1]}(IP2) + x^N_{[2]}(IP2) + x^N_{[3]}(IP2) = 7$. Then, he will choose $x = 7 \in X_{[4]}$, $\sum_i x^N_i(IP2) = 14$ (see Table 2) and $\sum_i C^i = \max\{4, 8\} + \max\{7, 11\} + 18 + \max\{14, 16\} = 53$.

6.3 Strategies for IP3

The player strategies for combination P/IP3 were verified in Example 2. For N/IP3 every player will assume that the portion $P_i - p^i_{\max}$ of the workload of every other player is infinitely divisible and that every other player will outsource precisely x^P_i units of work. According to IRN, every player has to outsource jobs from a knapsack of size no less than any of his predecessors. Therefore, players $[1], [2], \ldots, |M|-1$ will stick to strategies $f_{[1]}(x^P_{[1]}), f_{[2]}(x^P_{[2]}), \ldots, f_{[|M|-1]}(x^P_{[|M|-1]})$. As in the previous subsection, if the player scheduled last is known (as is the case when a single player attains the maximum workload), he can make a more educated decision by taking advantage of Lemma 4(a). As in the previous subsection, he will choose $x^\star \in X_{[|M|]}$ that solves

$$\min_{x \in X_{[|M|]}} \max\{P_{[|M|]} - x, \sum_{j=1}^{|M|-1} x^P_{[j]} + x\}$$

subject to constraints $\sum_{j=1}^{|M|-1} x^P_{[j]} + x < P_{|M|} - f_{|M|,|N_{[|M|]}|}(x, l(x)) + p_{[|M|],l(x)}$ which takes $\mathcal{O}(|N_{|M|}|^3 |M| P_{|M|})$ time. In our example, player $[4]$ will assume that the preceding players will outsource $x^P_{[1]} + x^P_{[2]} + x^P_{[3]} = 5 + 3 + 5 = 13$ units which is in fact true because $x^N_{[1]}(IP3) + x^N_{[2]}(IP3) + x^N_{[3]}(IP3) = 5 + 3 + 5 = 13$. Then, he will choose $x = 7 \in X_{[4]}$, $\sum_i x^N_i(IP3) = 20$

(see Table 2) and $\sum_i C^i = \max\{5,7\} + \max\{8,11\} + \max\{13,13\} + \max\{20,16\} = 51$.

6.4 Centralized Strategies

It is interesting to compare equilibrium schedules against their centralized counterparts. We assume that the central decision maker can choose a strategy on behalf of $i \in M$ so as to maximize $\sum_i x_i$. However, when overlapping or preemption is allowed the strategy of every player i must be such that his makespan is attained on M_i. This is due to Theorem 1 and Lemma 2(b) respectively. In contrast, for non-preemptive schedules Lemma 4(a) must hold for every player. These constraints are necessary to ensure that no player increases his makespan by outsourcing more than he should. Solving the 3 centralized problems (overlapping, preemptive and non-preemptive) is non-trivial and is beyond the scope of this article. However, this can be done by inspection for our example problem. Let $IP5$ be the centralized information protocol and $x_i^O(IP5), x_i^P(IP5), x_i^N(IP5)$ the associated workloads for $i \in M$.

Example 4: It is intuitive that the centralized strategy when overlapping is allowed is to schedule the 4 players in nondecreasing order of P_i and let an earlier player outsource as much as he productively can. Then, $x_1^O(IP5) = \frac{12}{2} = 6$, $x_2^O(IP5) = \frac{14-6}{2} = 4$, $x_3^O(IP5) = \frac{18-10}{2} = 4$ and $x_4^O(IP5) = \frac{23-14}{2} = 4.5$. Then, $\sum_i x_i^O(IP5) = 18.5$ (see Table 2) and $\sum_i C^i = 6+10+14+18.5 = 48.5 = \sum_i P_i - \sum_i x_i^O(IP5)$. The centralized optimum for preemptive scheduling is obtained similarly except that player 2 outsources only $P_3 - p^3_{\max} = 3$ units of work. The resulting solution is $x_1^P(IP5) = 6$, $x_2^P(IP5) = 3$, $x_3^P(IP5) = 4.5$ and $x_4^O(IP5) = 4.75$. Then, $\sum_i x_i^P(IP5) = 18.25$ and $\sum_i C^i = 6 + 11 + 13.5 + 18.25 = 48.75 = \sum_i P_i - \sum_i x_i^P(IP5)$. By inspection, strategies $x_i^N : i \in M$ provide an optimal centralized solution and hence $\sum_i x_i^N(IP5) = 20$ and $\sum_i C^i = 6 + 11 + 13.5 + 18.25 = 51 \neq \sum_i P_i - \sum_i x_i^N$.

7 Managerial Insights

In this section we summarize our findings based on the theoretical results presented in previous sections and our illustrative example. Both highlight that

- Increased information sharing is beneficial to P and to M as a whole.

The fact that better information results to better performance for M and P is obvious from a theoretical standpoint because additional information allows players to predict more accurately the contribution of other manufacturers to F and hence outsource more of their workload without fear of been delayed at F. For our example, this is evident in the last 2 rows of Table 2 where for all O, P and N, gains increase as we move from IP1 to complete information.

It is interesting to examine the above finding in the context of individual players because information cannot be shared without their consent. Indeed, we observe that

- Increased information sharing is beneficial to all players.

This is evident from Table 1 where for all O, P and N, each player outsources at least as much as in the preceding information protocol. Note that this includes player $|M|$ in combinations P/IP2 and N/IP3 who optimizes so as to satisfy Lemma 4(a). However, our observation does not necessarily hold for N/IP4 because every player takes complete advantage of small workloads outsourced by players preceding him. As a result, early players outsource more, and hence subsequent players (and specifically player $|M|$) are left with less useful capacity on F and may be forced to outsource less under IP4 even though they may be drawing from a knapsack as large as their immediate predecessor. In Table 2 however, all players benefit with more information sharing.

Both insights reported so far are in contrast to what has been reported in the supply chain management literature when retailers share information vertically with a manufacturer or horizontally amongst them. For example, Li (2002) reports that horizontal information sharing discourages retailers from sharing their demand information with the manufacturer while encourages them to share their cost information. However, the production settings considered in this article and in Li (2002) are not comparable.

7.1 Marginal Information Gains

We now turn our attention to marginal gains due to information. Given a production protocol O, P, or N, define the marginal value of information (VoI) to M (denoted by V_{12}^M, V_{23}^M, V_{34}^M, V_{45}^M respectively) as the marginal total completion time savings between protocols IP1,IP2 between IP2,IP3, between IP3,IP4 and IP4,IP5 respectively. Similarly, given a production protocol among O,P,N, define the marginal value of information to P (denoted by V_{12}^P, V_{23}^P, V_{34}^P, V_{45}^P respectively) as the marginal increase in total outsourced workload between protocols IP1,IP2 between IP2,IP3, between IP3,IP4 and between IP4,IP5. For a given production protocol O,P,N, let $C^i(IP1), C^i(IP2), C^i(IP3), C^i(IP4), C^i(IP5)$, $i \in M$ be the makespan of the manufacturers in the equilibrium schedules for protocols IP1-IP4 and the centralized schedule for IP5 respectively. Then,

$$\begin{aligned}
V_{12}^M &= \sum_{i=1}^{|M|}(C^i(IP1) - C^i(IP2)), \qquad & V_{12}^P &= \sum_{i=1}^{|M|}(x_i(IP2) - x_i(IP1)),\\
V_{23}^M &= \sum_{i=1}^{|M|}(C^i(IP2) - C^i(IP3)), \qquad & V_{23}^P &= \sum_{i=1}^{|M|}(x_i(IP3) - x_i(IP2)),\\
V_{34}^M &= \sum_{i=1}^{|M|}(C^i(IP3) - C^i(IP4)), \qquad & V_{34}^P &= \sum_{i=1}^{|M|}(x_i(IP4) - x_i(IP3)),\\
V_{45}^M &= \sum_{i=1}^{|M|}(C^i(IP4) - C^i(IP5)), \qquad & V_{45}^P &= \sum_{i=1}^{|M|}(x_i(IP5) - x_i(IP4)).
\end{aligned}$$

For production protocols O and P, Theorem 1 and Lemma 2(b) imply that $C^i(IP) = P_i - x_i$ for $i \in M$ and $IP \in \{IP1, IP2, IP3, IP4, IP5\}$. Hence, for protocols O and P we have $V_{12}^M = V_{12}^P$, $V_{23}^M = V_{23}^P$, $V_{34}^M = V_{34}^P$ and $V_{45}^M = V_{45}^P$ and for this reason in Table 3 we report a single value for every O/IP and P/IP combination corresponding to the strategies in Table 2.

Table 3: Marginal gains for M and P due to information

VoI	$V_{1,2}^M, V_{1,2}^P$	$V_{2,3}^M, V_{2,3}^P$	$V_{3,4}^M, V_{3,4}^P$	$V_{4,5}^M, V_{4,5}^P$
O	4.85	0	0	0.25
P	3.35	1.25	0	0.25
N	10	2, 6	0, 0	0, 0

For protocol N, expression $C^i(IP1) = P_i - x_i$ holds because every player outsources no more than he would if overlapping was allowed (see Table 1). For $IP = IP2, IP3$ expression $C^i(IP) = P_i - x_i$ holds for all players except possibly player $|M|$, for similar reasons. However, for $IP = IP4, IP5$ we have $C^i(IP) \geq P_i - x_i$. Therefore, for protocol N we have $V_{12}^M = V_{12}^P$, $V_{23}^M \leq V_{23}^P$, $V_{34}^M \leq V_{34}^P$ and $V_{45}^M \leq V_{45}^P$. For this reason combination N/IP1 is presented in Table 3 with a single value. We can now make the following observation.

- For M as a whole and for P, the gains due to information sharing increase at a decreasing rate.

The fact that these gains are decreasing in the amount of information is certainly evident from Table 3 for our example problem where the values associated with the protocols IP1-IP4 decrease. To further validate our claim we performed a computational experiment for all $(|N_i|, |M|)$ combinations indicated in Table 4, information protocols IP1-IP4 and all 3 production protocols. For each combination we randomly generate 10 problems with processing times drawn from a discrete uniform distribution $U = [1, 5]$ or $U = [1, 10]$. For each problem we compute the values P_i and p_{max}^i, $i \in M$ and subsequently find the Nash strategies for every protocol O, P, N. For given $|N_i|$, $|M|$, the same 10 problems are considered under all 3 production protocols.

In Table 4 we report the average percentage marginal VoI computed as

$$\hat{V_z^M} = \frac{\sum_{i=1}^{|M|}(C^i(z) - C^i(IPc))}{\sum_{i=1}^{|M|} C^i(IPc)} 100\%, \quad \hat{V_z^P} = \frac{\sum_{i=1}^{|M|}(x_i(IPc) - x_i(z))}{\sum_{i=1}^{|M|} x_i(IPc)} 100\%,$$

where $z \in \{IP1, IP2, IP3\}$ and IPc refers to complete information. For protocol N, solving the centralized problem to maximize the total outsourced workload has not been considered in the literature and is beyond the scope of this article. Therefore, in this experiment we do not report values $\hat{V}_4^M$ and $\hat{V}_4^P$ for protocols O,P and N (also note that values $\hat{V}_3^M$, $\hat{V}_3^P$ are null for O,P). All remaining values can be computed based on the optimal Nash strategies presented in the previous sections and the fact that for protocols O and P maximizing the total outsourced workload is congruent to minimizing the total flowtime of the players.

Table 4: Percentage deviations $\hat{V}_z^M$, $\hat{V}_z^P$

			O / P		N	
$\lvert M\rvert$	$\lvert N_i\rvert$	p_j	$\hat{V}_1^M, \hat{V}_1^P$	$\hat{V}_2^M, \hat{V}_2^P$	$\hat{V}_1^M, \hat{V}_1^P$	$\hat{V}_2^M, \hat{V}_2^P$
4	5	[1,5]	22.32	0	32.43	9.32
		[1,10]	21.73	0	33.51	12.82
	10	[1,5]	18.95	0	22.35	10.67
		[1,10]	19.76	0	22.04	9.53
	15	[1,5]	18.58	0	19.91	11.90
		[1,10]	18.80	0	19.79	9.44
6	5	[1,5]	24.19	0	50.30	22.74
		[1,10]	24.00	0	46.83	19.97
	10	[1,5]	20.26	0	26.23	12.59
		[1,10]	21.66	0	26.05	9.12
	15	[1,5]	19.47	0	22.69	11.60
		[1,10]	18.23	0	22.12	8.32
8	5	[1,5]	25.16	0	56.01	18.83
		[1,10]	24.78	0	61.02	28.85
	10	[1,5]	21.06	0	29.27	12.65
		[1,10]	21.66	0	26.47	10.28
	15	[1,5]	19.07	0	24.24	11.45
		[1,10]	18.98	0	23.79	9.85
Averages			21.04	0	31.26	14.16

For all our problems the total outsourced amount was the same for protocols $IP1$, $IP2$ and $IP3$ for all O, P and N. This is why $\hat{V}_2^M = \hat{V}_2^P = 0$ for O, P in Table 4 and for the same reason we consolidated the columns corresponding to protocols O and P. This indicates that preemption adds to problem difficulty only when some players have to process a really big job that is greater than half their total workload, as in our example problem. For protocol N, the fact that preemption resulted to the same solutions as overlapping means that $\hat{V}_2^M = \hat{V}_3^M$ and $\hat{V}_2^P = \hat{V}_3^P$ for N and hence in our table we simply report $\hat{V}_2^M, \hat{V}_2^P$. Moreover, with our randomly generated problems we obtained $\hat{V}_2^M = \hat{V}_2^P$ with all instances.

The following observations are made from Table 4. First, the marginal value of information decreases as players reveal more information about their processing profiles. This is clearly demonstrated with every one of the parameter combinations tested. Moreover, the VoI is

- greater for N than for O, P,

- increases with the number $|M|$ of players,

- insensitive to the processing time of the jobs of the players.

All of these observations hold true for all parameter combinations in Table 4. In conclusion, all our findings collaborate in that all members of the production chain can benefit by information sharing. In our example problem, information sharing provides an incentive to all for near optimal system performance without centralized control and is of tremendous importance because centralized control requires an additional layer of management which is costly and impedes manufacturers behaviorally.

8 Conclusions

In this article we considered a competitive model of outsourcing operations to a third-party so as to take best advantage of her capacity. In the first part of the article we developed pure equilibria strategies for 3 production and 4 information protocols. Subsequently, we studied the role of information on player strategies and found that information sharing is beneficial to all in the production chain. Our findings led to the observation that near optimal performance can be achieved without the burden of centralized control if P chooses an appropriate incentive rule and the manufacturers share sufficient amounts of information about their job profiles.

A number of future research directions are possible beyond the countless variations in production configurations, player objectives (including cost rather than time-based functions) and incentive rules. Another obvious research direction is the analysis of the centralized problem presented in this article and the consideration of a cooperative game that helps players achieve the results of coordination by means of transfer payment schemes instead of rules imposed by the third party. In our article we assumed that under incomplete information players respond with strategies that hedge against the worst possible outcome. Distributional assumptions for possible outcomes within a Bayesian framework may point to different interpretations of the value of information. Our immediate future research will focus on outsourcing models with transportation considerations because of the growing importance of international (out)sourcing.

References

[1] Aviv, Y.A. and Federgruen, A. The Operational Benefits of Information Sharing and Vendor Managed Inventory (VMI) Programs, Washington University, St. Louis, MO, 1998.

[2] Aydinliyim, T. and Vairaktarakis, G. L. Coordination of Outsourcing Operations, *Manufacturing & Service Operations Management*, submitted.

[3] Bartal, Y., Leonardi, S., Marchetti-Spaccamela, A., Sgall, J. and Stougie, L. Multiprocessor Scheduling with Rejection, in *Proceedings of the 7th ACM-SIAM Symposium on Discrete Algorithms*, pp. 95-103, 1996.

[4] Bell, C.E. and Stidham, S. Individual versus Social Optimization in the Allocation of Customers to Alternative Servers, *Management Science*, 43, 1609-1627, 1997.

[5] Bourland, K., Powell, S. and Pyke, D. Exploiting Timely Demand Information to Reduce Inventories, *European Journal of Operations Research*, 92, 239-253, 1996.

[6] Cachon, G.P. Supply Chain Coordination with Contracts. *in S. Graves, T. de Kok, eds. 'Handbooks in Operations Research and Management Science: Supply Chain Management'*, North-Holland, New York, NY, Chapter 6, 2003.

[7] Cachon, G.P. and Fisher, M. Supply Chain Inventory Management and the Value of Shared Information, *Management Science*, 46(8), 936-953, 2000.

[8] Cachon, G.P. and Lariviere, M.A. Capacity Choice and Allocation: Strategic Behavior and Supply chain Performance, *Management Science*, 45(8), 1091-1108, 1999.

[9] Chen, F. Echelon Reorder Points, Installation Reorder Points, and the Value of Centralized Demand Information, *Management Science*, 44, S221-S234, 1998.

[10] Curiel, I., Pederzoli, G. and Tijs, S. Sequencing Games, *European Journal of Operational Research*, 40, 344-351, 1989.

[11] Curiel, I., Hamers, H. and Klijn, F. Sequencing Games: A Survey, *in P. Borm and H. Peters, eds. 'Chapters in Game Theory: In Honor of Stef Tijs'* Kluwer Academic Publishers, Boston, pp. 27-50, 2002.

[12] Dewan, S. and Mendelson, H. User Delay Costs and Internal Pricing for a Service Facility, *Management Science*, 36, 1502-1517, 1990.

[13] Edelson, N.M. and Hildebrand, D.K. Congestion Tolls for Poisson Queuing Processes, *Econometrica*, 43, 81-92, 1975.

[14] Engels, D.W., Karger, D.R., Kolliopoulos, S.G., Sengupta, S., Uma, R.N. and Wein, J. Techniques for Scheduling with Rejection, *Journal of Algorithms*, 499, 175-191, 2003.

[15] Gavirneni, S., Kapuscinski, R. and Tayur, S. Value of Information in Capacitated Supply Chains, *Management Science*, 45, 16-24, 1999.

[16] Lee, H., So, K. and Tang, C. The value of Information Sharing in a two-level Supply Chain, *Management Science*, 46(5), 626-643, 2000.

[17] Hain, R. and Mitra, M. Simple Sequencing Problems with Interdependent Costs, *Games and Economic Behavior*, 48, 271-291, 2004.

[18] Hoogeveen, H. Preemptive Scheduling with Rejection, *Mathematical Programming*, 94:361-374, 2003.

[19] Kleinrock, L. Optimum bribing for Queue Position, *Operations Research*, 15, 304-318, 1967.

[20] Lariviere, M. and Porteus, E. Selling to the Newsvendor: An Analysis of Price-Only Contracts, *Manufacturing & Service Operations Management*, 3(4), 293-305, 2001.

[21] Lederer, P.J. and Li, L. Pricing, Production, Scheduling, and Delivery-time Competition, *Operations Research*, 45, 407-420, 1994.

[22] Lee, H., Padmanabhan, V. and Whang S. Information Distortion in a Supply Chain: The Bullwhip Effect, *Management Science*, 43, 546-558, 1997.

[23] Li, L. Information Sharing in a Supply Chain with Horizontal Competition, *Management Science*, 48(9), 1196-1212, 2002.

[24] Mendelson, H. and Whang, S. Optimal Incentive-Compatible Priority Pricing for the M/M/1 Queue, *Operations Research*, 38(5), 870-883, 1990.

[25] Nash, J.F. Non-Cooperative Games. *Annals of Mathematics*, 54, 286-295, 1951.

[26] Padmanabhan, V. and Png, I. Manufacturer's Return Policies and Retail Competition, *Marketing Science*, 16(1), 81-94, 1997.

[27] Pinedo, M. *Scheduling: Theory, Algorithms and Systems*, Prentice Hall, Englewood Cliffs, NJ, 2002.

[28] Seiden, S., Preemptive Multiprocessor Scheduling with Rejection, *Theoretical Computer Science*, submitted, 2000.

[29] Sengupta, S. Algorithms and Approximation Schemes for Minimum Lateness and Tardiness Scheduling with Rejection, Manuscript, Laboratory for Computer Science, MIT, 1999.

[30] Shapley, L. Cores of Convex Games, *International Journal of Game Theory*, 1, 11-26, 1971.