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# Coordination of Inventory, Capital and Dividends in Nascent Firms

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**Abstract**

Poor management of inventory and credit are often the culprits when nascent firms struggle. We consider how to coordinate borrowing and inventory management decisions in firms whose size changes over time. The firm in the model has a single location and single product for which there is an autoregressive demand process, and it optimizes the expected present value of the time stream of dividends net of capital subscriptions. The results include a myopic policy with coordinated base-stock levels for physical inventory and retained earnings. These levels change with the size of the firm, and we compare them with their counterparts when the firm decentralizes inventory and financial decisions. We find repeatedly that the relative improvement of coordination is greater for small firms than for large.

*Key words:* Coordination, Inventory, Dividends, Myopic policy, Nonstationary demand

## 1 Introduction

Poor management of inventory and credit are often the culprits when nascent firms struggle (e.g. Ames (1983)). The U.S. Small Business Administration emphasizes the importance of successfully managing financial resources in newly created firms, and observes (sba.gov (2007)) that a nascent firm can improve its financial base either by growing gradually and allowing profits to fund additional growth, or by seeking outside funds such as short-term loans or lines of credit. However, it is

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unclear whether firms should borrow early on, or whether they should delay borrowing while they build equity through retained earnings. In addition to managing access to capital and liquidity, firms must contain their costs. Effective inventory management is an important element of cost containment, particularly in retail and manufacturing firms, but the commonly utilized inventory models are divorced from financial considerations. The models implicitly assume that the theorem of Modigliani and Miller (1958) is valid, although the premises of this theorem are frequently invalid due to factors such as incipient bankruptcy costs.

This paper considers how best to coordinate borrowing and inventory management decisions in recently formed corporate firms. Leveraged firms are extensively analyzed in the finance literature. Merton (1974) considers a static problem of pricing corporate debt when there is a positive probability that the firm will be unable to satisfy some of its debt obligations. Although that research has been extended to multi-period formulations by Geske (1977) and others, the results offer little guidance for non-financial decisions. The Black-Scholes framework introduced in the Merton model is well suited for valuation and hedging, and these are foci of corporate finance, but it is not well suited for optimizing production and related decisions. Although the valuation problem is well understood, there has been little analysis of optimal operating policies when firms have a significant probability of default. Two exceptions are Xu and Birge (2004) and Li et al. (1997) who study the coordination of operating and financial decisions, but the assumptions in both works preclude firm growth. Here we consider the coordination of operating and financial decisions when changes in demand may occur due to growth, seasonality, or other factors.

Answers to questions such as ours are driven by model details. Since we seek to obtain as much insight into coordination as possible, we use simple operational and financial structures in the model and an autoregressive demand process. This type of process is nonstationary but tractable. On the operational side, the firm produces a single product at a single location for a market with an exogeneously fixed price. The firm has 100% yields, there is no delay in the availability of goods to satisfy demand, and it experiences linear costs of production and storage. On the financial side, the firm can augment its working capital only with short-term borrowing and capital subscriptions. At first we model financial distress with the costs of reorganization that follow bankruptcy. This assumption is based on data that show that the majority of defaults do not result in liquidation (Franks and Torrous (1994)). Instead, the defaulting firm is often restructured and creditors are compensated with a new debt issue. Later we consider the opposite extreme, namely dissolution of the firm following bankruptcy.

The most important measure of the value of an investor-owned firm is the expected present value of the sequence of its dividends (cf. Brealey and Myers (2006), Cochrane (2001), and Ross et al. (2006)). So the optimization criterion in this paper is the expected present value of the dividends net of capital subscriptions. Capital subscriptions (i.e., additional capital raised from the current investors) occur often in nascent firms.

We make no attempt to assign a market value to the firm in the model and, as a consequence, we do not specify a pricing measure. Our discount factor is a placeholder for what would otherwise be a risk-free discount factor under the risk-neutral pricing measure (or an expected value of the pricing kernel under the real-world measure). This shortcut does not affect the conclusions but it streamlines the exposition.

The results include the complete characterization of an optimal policy that is myopic and has base-stock levels for physical goods inventory and retained earnings. The physical inventory base-stock level is lower than that in the corresponding standard inventory model divorced from financial considerations. The optimal short-term loan follows the well-known pecking-order hypothesis. We compare the effects of optimal coordinated decisions with the effects of decentralized decisions. We find repeatedly that the relative improvement of coordination is greater for small firms than for large. That is, the application of standard inventory and production models may be particularly ill-advised in small firms.

Our proxy for firm size is the deterministic part of the autoregressive demand process, $\mu$. Let $r$ denote the exogeneous market price for the firm's good. The inventory base-stock level equals $\mu$ plus a factor, and the one for retained earnings equals $-r\mu$ plus a factor. The size of the resulting short-term loan is $r\mu$ plus a factor. None of these factors depend on $\mu$, and the optimal policy induces dividends that grow with firm size at the rate of $r(1-\rho)$ where $\rho$ is the interest rate for short-term borrowing.

It is important to examine the dependence of the conclusions on the assumptions. For example, bankruptcy is a complex financial and legal process, and in most of the paper the firm's inability to repay a loan leads to a default penalty, namely the costs of restructuring debt. However, §7 explains why the replacement of the restructuring assumption with irrevocable dissolution of the firm would preserve key results. Although we introduce a general autoregresssive demand process in §3, thereafter we employ a first-order process for expository simplicity. Most of the paper's results would be preserved with a higher-order process. In §5 we introduce piece-wise linear expressions for the default penalty and the sales revenue net of inventory-related costs. Most of the qualitative results would be preserved with more general functions but simple explicit formulas would be lost.

The paper is organized as follows. The model is specified in §3, and leads in §4 to properties of a relevant dynamic program and the structure of an optimal policy. A myopic policy is shown to be optimal in §5, and the piece-wise linearity assumption yields simple explicit formulas. Some consequences of firm size are analyzed in §6 which also compares the myopic optimal policy with a decentralized policy. In §7 the myopic policy is shown to induce a probability of default that is constant with respect to firm size, and the model is changed to examine the consequences if bankruptcy causes irrevocable dissolution of the firm. The results are summarized in §8 which briefly discusses the effects of altering the assumptions and model details.

# 2 Related Research

There are several sub-literatures on the coordination of operations and finance. They include operational decisions in the presence of foreign exchange exposure (e.g. Kogut and Kulatilaka (1994), Huchzermeier and Cohen (1996), Dasu and Li (1997), Aytekin and Birge (2004), Dong et al. (2006)) and capacity-expansion problems with financial constraints (e.g. Birge (2000), Van Mieghem (2003), and Babich and Sobel (2004)). Some of that research uses approaches similar to ours, but this paper contributes to the sub-literature on the coordination of production and financing decisions and is closest to Li et al. (1997). Their model is the special case of ours with a $0^{\text{th}}$-order autoregressive demand process, namely independent and identically distributed random demands. Also, excess demand is backordered in their paper and lost in ours. The authors find that a myopic policy is optimal and study its properties.

Other papers based on Li et al. (1997) are Hu and Sobel (2005) and Brunet and Babich (2006). The former examines the impact of a firm's capital structure on its short-term operating and financial decisions. The latter studies the value of trade credit when the firm's borrowing capacity is uncertain.

Earlier contributions to the coordination of production and financial decisions include Archibald et al. (2004) and Xu and Birge (2004). The former optimizes the probability of financial survival of a start-up firm that manages inventory, but does not seek outside funds. The analysis stems from the hypothesis that nascent firms are more concerned with long-term survival than with profitability. The latter uses a one-period newsvendor framework to show that the firm's optimal production quantity is a decreasing function of its financial leverage and that misidentifying the company's optimal leverage ratio decreases the firm's market value.

The important difference between these models and ours is that they preclude the possibility of firm growth because they either assume identically distributed demands or they are one-period models. A more general demand model is needed here because growth is the chief motivation of the nascent firm that we analyze. Our use of a non-stationary demand process leads to insights that cannot be elicited from models with stationary or static demands.

# 3 The Model

We use a discrete-time multi-period model of a single-product single-location corporate firm that decides at the beginning of each period how much money to borrow, $b_n$, how many units to produce or procure, $z_n$, and how much of a dividend to issue, $v_n$. Then a stochastic demand, $D_n$, occurs and various revenues are received and costs paid.

At the beginning of each period $n$, $(n = 1, 2, ...)$ the firm observes the amount of retained earnings, $w_n$ (unconstrained in sign), the current inventory level, $x_n \geq 0$ (implying that excess demand is lost), and the amount demand in the previous period, $D_{n-1}$. If $w_n < 0$, then the firm was unable to meet some of its debt obligations in period $n-1$ and it is assumed to be in bankruptcy at the start of period $n$. When in bankruptcy, the firm is assessed a default penalty $p(w_n)$, but it does not cease operations. Actual bankruptcy processes can be quite complex. Therefore, for modeling purposes, we make the simplifying assumption of debt refinancing. In the model, the restructuring costs are represented by the default penalty $p(w_n)$. In §7, we replace restructuring with the complete cessation of operations.

Prior to observing demand in period $n$ demand, the firm makes these decisions: $z_n \geq 0$, $b_n \geq 0$, and $v_n$ (unconstrained in sign). We interpret negative values of $v_n$ as capital subscriptions which occur often among entrepreneurial ventures.

Production and borrowing costs are $cz_n$ and $\rho b_n$, respectively, where $c$ and $\rho$ denote the unit costs of production and borrowing. We assume that the production lead time is negligible; so $x_n + z_n$ is the supply of goods that are available to satisfy the demand in period $n$. Borrowing is short-term; that is, the outstanding principal, $b_n$, is due to be repaid at the end of period $n$. We assume that the borrowing cost rate $\rho$ is a function of the default probability, but that probability turns out to be constant (cf. §7). So we interpret $\rho$ as the value taken by the function at the constant default probability.

We assume that the sales revenue net of inventory-related costs, denoted $g(x_n+z_n, D_n)$, depends on the total supply and demand in period $n$.

The flow of goods and dollars in the model is subject to *conservation* constraints

$$w_{n+1} = w_n - p(w_n) - v_n - cz_n + g(x_n + z_n, D_n) - \rho b_n, \tag{1}$$

$$x_{n+1} = (x_n + z_n - D_n)^+,$$

a *liquidity* constraint that prevents the expenditures in period $n$ from exceeding the sum of retained earnings plus the loan proceeds

$$w_n + (1 - \rho)b_n \geq cz_n + v_n + p(w_n), \tag{2}$$

and *logical* constraints

$$b_n \geq 0 \qquad\qquad \text{and} \qquad\qquad z_n \geq 0. \tag{3}$$

By the end of period $n$, the firm will have observed demand $D_n$, realized revenue net of inventory costs $g(x_n + z_n, D_n)$, and repaid the entire loan principal $b_n$ if $w_{n+1} \geq 0$, where $w_{n+1}$ is given by (1). Otherwise, there is a delay in repayment and the default penalty $p(w_{n+1})$ is levied.

## Demand Model

An autoregressive processes makes it possible for firm size to change over time. These processes are attractive too because they are well understood (c.f. Greene (2003), Hamilton (1994)), relatively easily estimated, and encompass many forms of systematic non-stationarity, such as seasonality and trends. Therefore, let

$$D_n = \sum_{k=1}^{K} \theta_k D_{n-k} + \varepsilon_n \tag{4}$$

where $\{\theta_k\}$ are known scalars, and $\{\varepsilon_n\}$ are independent, identically distributed, and nonnegative random variables.

Let $\varepsilon$ be a random variable with the same distribution as $\varepsilon_1$, and let $F(\cdot)$ denote the distribution function of $\varepsilon$. We say that $q^*$ *solves* $F(q^*) = \alpha$ if $q^* = \sup\{q : F(q) \leq \alpha\}$, and we write $q^* = F^{-1}(\alpha)$. Similarly, we say that $q^*$ solves $\lambda_1 F(q^*) + \lambda_2 F(\lambda_3 q^*) = \alpha$ if $q^* = \sup\{q : \lambda_1 F(q) + \lambda_2 F(\lambda_3 q) \leq \alpha\}$ (where each $\lambda_j \geq 0$ and $0 < \alpha < 1$). This convention avoids the need to assume that $\varepsilon$ has a density function.

Henceforth, we let $K = 1$ for expository simplicity, although all results are valid if $K > 1$. The special case $K = 0$ yields a sequence of demands that are independent and identically distributed. The assumption $K = 1$ and the notation $\mu_n = \theta D_{n-1}$ reduces (4) to the following demand model used in the remainder of the paper:

$$D_n = \theta D_{n-1} + \varepsilon_n = \mu_n + \varepsilon_n \tag{5}$$

We interpret $\mu_n$ as the *deterministic part of demand* in period $n$.

### Optimization Objective and Dynamic Program

The following redefinition of the decision variables shortly reduces the dimensionality of the state vector. The idea is to replace the flow variables $z_n$ and $v_n$ with new variables that specify process levels after the period $n$ decisions are implemented. Let

$$y_n = x_n + z_n \tag{6a}$$

$$s_n = w_n - p(w_n) - v_n - cz_n - \rho b_n \tag{6b}$$

The procurement/production quantity, $z_n$, is replaced by the supply level $y_n$ and the dividend, $v_n$, is replaced by $s_n$. The latter is the working capital after paying the dividend, loan interest, and production costs, but before the loan is made and the revenue and inventory costs are realized.

This replacement yields a simple form of the liquidity and logical constraints (2) and (3):

$$b_n + s_n \geq 0, \qquad b_n \geq 0, \qquad \text{and} \qquad y_n \geq x_n \tag{7}$$

Let the scalar $\beta$ denote the single-period discount factor ($0 < \beta < 1$). A constant discount factor streamlines the exposition and does not affect the validity of most results (see §7).

The present value of the dividends net of capital subscriptions is

$$B = \sum_{n=1}^{\infty} \beta^{n-1} v_n, \tag{8}$$

Let $H_n$ denote the partial history from period one up to the beginning of period $n$. A *policy* is a non-anticipative decision rule that selects $z_n$, $b_n$, and $v_n$ for each $n$ as a function of $H_n$. A policy is *optimal with respect to the set of initial states* $S$ if it maximizes $E(B|H_n)$ for each $H_n$ and $n$ such that $(x_1, \mu_1) \in S$. In the remainder of the paper we characterize an optimal policy.

Substituting $v_n = w_n - p(w_n) - s_n - c(y_n - x_n) - \rho b_n$ (from (6b)), $x_{n+1} = (y_n - D_n)^+$ (from (3)), and $w_n = s_{n-1} + g(y_{n-1}, \mu_{n-1} + \varepsilon_{n-1})$ in (8), rearranging terms, and defining

$$L(b, s, y, \mu) = -(1-\beta)s - cy + \beta c E[(y - \mu - \varepsilon)^+] + \beta E\{g(y, \mu + \varepsilon) - p[s + g(y, \mu + \varepsilon)]\} - \rho b \tag{9}$$

yields

$$E(B) = cx_1 + w_1 - p(w_1) + E \sum_{n=1}^{\infty} \beta^{n-1} L(b_n, s_n, y_n, \mu_n) \tag{10}$$

Since a policy maximizes $E(B)$ if and only if it maximizes $E(B') = E(B) - (cx_1 + w_1 - p(w_1)$, it follows that the maximization of $E(B)$ subject to (2) and (3) corresponds to the optimization of

$$E(B') = E \sum_{n=1}^{\infty} \beta^{n-1} L(b_n, s_n, y_n, \mu_n) \tag{11}$$

with the following constraints for each $n$:

$$b_n + s_n \geq 0, \qquad b_n \geq 0, \qquad \text{and} \qquad y_n \geq x_n \tag{12}$$

This optimization of the expected present value of dividends corresponds to the following dynamic program:

$$\psi(x, \mu) = \sup_{b,s,y} \{J(b, s, y, \mu) : y \geq x, b \geq 0, b + s \geq 0\} \tag{13a}$$

$$J(b, s, y, \mu) = L(b, s, y, \mu) + \beta E\{\psi[(y - \mu - \varepsilon)^+, \theta(\mu + \varepsilon)]\} \tag{13b}$$

A finite-horizon recursion that corresponds to (13) is $\psi_0(\cdot, \cdot) \equiv 0$ and

$$\psi_n(x, \mu) = \sup_{b,s,y} \{J_n(b, s, y, \mu) : y \geq x, b \geq 0, b + s \geq 0\} \tag{14a}$$

$$J_n(b, s, y, \mu) = L(b, s, y, \mu) + \beta E\{\psi_{n-1}[(y - \mu - \varepsilon)^+, \theta(\mu + \varepsilon)]\} \tag{14b}$$

for each $n = 1, 2, \ldots$. Under reasonable conditions the sequence $\psi_n$ converges pointwise to $\psi$ as $n \to \infty$.

# 4 Preliminary Results

The first result asserts that it is optimal to borrow the smallest amount that satisfies the liquidity constraint (7). The proof (omitted) is a straightforward generalization of a proof of a similar result in Li et al. (1997).

**Proposition 1.** *In (14), $b = (-s)^+$ is without loss of optimality.*

This result is consistent with the well-known pecking-order hypothesis, which says that firms use internally generated capital before turning to more expensive sources of financing. After internal sources are exhausted, firms will resort to debt.

The following result asserts that the dynamic program value function is a concave function. We omit the inductive proof which starts with $\psi_0(\cdot, \cdot) \equiv 0$.

**Proposition 2.** *If $p(\cdot)$ is convex and $g(\cdot, d)$ is concave (for each possible realization of demand $d$), then the dynamic program value function $\psi_n(\cdot, \cdot)$ in (14a) and $J_n(\cdot, \cdot, \cdot, \cdot)$ in (14b) are concave functions on their respective domains ($n = 1, 2, ...$).*

Let $(b, s, y) = (b_n(\mu), s_n(\mu), y_n(\mu))$ optimize $J_n(b, s, y, \mu)$ in (14) and (14a) subject only to $b \geq 0$ and $b+s \geq 0$ (i.e., *not* subject to $y \geq x$). Concavity yields the following result that is a consequence of Propositions 1 and 2 (we omit the proof).

**Proposition 3.** *Under the hypotheses of Proposition 2, $(b, s, y)$=$((-s_n(\mu))^+$,$s_n(\mu)$,$\max\{y_n(\mu), x\})$ is optimal in (14).*

Proposition 3 says that optimal behavior consists of borrowing the smallest amount that satisfies the liquidity constraint and producing nothing if the inventory on hand exceeds the optimal base-stock level.

# 5 A Myopic Optimum

Henceforth, we assume that the default penalty function and the net-revenue function are piece-wise linear:

$$p(x) = (-ax)^+ \quad (a \geq 0) \tag{15}$$

$$g(y, d) = ry - (r + h)(y - d)^+ \tag{16}$$

Here, $r$ and $h$ are an exogeneous selling price and a holding cost rate. Notice that (16) is consistent with the assumption that excess demand is lost. The piece-wise linearity in (15) and (16) is not necessary for our qualitative conclusions, but it yields simple explicit formulas.

The substitution of (15), (16), and $b = (-s)^+$ in (9) yields

$$\begin{aligned} L(b, s, y, \mu) \quad &= \quad -(1-\beta)s - \rho(-s)^+ + (\beta r - c)y - \beta(r + h - c)E[(y - \mu - \varepsilon)^+] \\ &- \quad \beta a E\{[(r + h)(y - \mu - \varepsilon)^+ - s - ry]^+\} \end{aligned} \tag{17}$$

We use the mnemonic $C$ for "coordinated." Let $(b,s,y) = (b^C(\mu), s^C(\mu), y^C(\mu))$ maximize $L(b,s,y,\mu)$ subject to $b+s \geq 0$ and $b \geq 0$. We focus on a myopic policy $\pi^C$ that assigns $(b_n,s_n,y_n) = (b^C(\mu_n), s^C(\mu_n), y^C(\mu_n))$ if this triple if feasible, i.e. if $x_n \leq y^C(\mu_n)$, and assigns $(b_n,s_n,y_n)$ arbitrarily but feasibly if $y^C(\mu_n) < x_n$ $(n = 1,2,...)$. The attraction of $\pi^C$ is that it would indeed be optimal with respect to $\{(x,\mu) : x \leq y^C(\mu)\}$ if $x_1 \leq y^C(\mu_1)$ and $\pi^C$ induces $x_n \leq y^C(\mu_n)$ for all $n$. In accordance with Proposition 1, henceforth we assign $b^C(\mu) = (-s^C(\mu))^+$.

If the initial inventory were low enough (i.e. $x_1 \leq y^C(\mu_1)$), then $(b_1,s_1,y_1) = ((-s^C(\mu_1))^+, s^C(\mu_1), y^C(\mu_1))$ would be feasible in period one. If those decisions were actually executed, then a sufficient condition for $(b_2,s_2,y_2) = ((-s^C(\mu_2))^+, s^C(\mu_2), y^C(\mu_2))$ to be feasible in period two would be that $\{\mu_n\}$ is non-decreasing over time (with probability one). This is unreasonably restrictive and *we do not make this assumption.* We focus on $\pi^C$ because a much weaker assumption implies feasibility, hence optimality, of $(b_n,s_n,y_n) = ((-s^C(\mu_n))^+, s^C(\mu_n), y^C(\mu_n))$ for all $n = 1,2,\ldots$

Let $y^*$ be the largest value of $y$ that satisfies

$$(r+h-c)F(y) + ahF\left(\frac{hy}{r+h}\right) \leq r - c/\beta \tag{18}$$

We make the following assumption throughout the remainder of the body of the paper.

$$F\left(\frac{hy^*}{r+h}\right) < \frac{1-\beta-\rho}{\beta a} < \frac{\beta r - (1-\beta-\rho)h - c}{\beta(r+h-c)} \tag{19}$$

The rationale for (19) begins with Proposition 2 which implies that $(b,s,y) = (b^C(\mu),\ s^C(\mu),\ y^C(\mu))$ maximizes a concave function of three variables subject to two constraints. So one route to a complete solution is the application of the Karush-Kuhn-Tucker conditions to the optimization problem. This route is followed in Appendix A and results in a partition of the set of all possible problem parameters into five subsets. The relevant parameters are $\beta, \rho, c, r, h, a$, and the distribution function $F(\cdot)$ of the random portion of demand $\varepsilon$. The body of this paper is confined to the subset that we judge most likely to be encountered in practice, namely (19). The complete solution is given in Appendix A.

Condition (19) is stronger than necessary for the next result, but it simplifies the exposition. Following Proposition 5 and in Appendix A.3 there are discussions of the restrictions imposed by (19).

An optimal coordinated policy is myopic and stipulates base-stock levels for physical inventory and retained earnings. Let

$$y^C(\mu) = \mu + f_0 \qquad f_0 = F^{-1}\left[\frac{\beta r - (1-\beta-\rho)h - c}{\beta(r+h-c)}\right] \tag{20}$$

$$s^C(\mu) = hf_0 - (r+h)f_1 - r\mu \qquad f_1 = F^{-1}\left[\frac{1-\beta-\rho}{\beta a}\right] \tag{21}$$

**Proposition 4.** *Assumption (19) implies that* $\pi^C$ *is optimal with respect to* $\{(x, \mu) : x \leq y^C(\mu)\}$. *That is, if* $x_1 \leq y(\mu_1)$ *then* $(b, s, y) = ((-s^C(\mu_n))^+, s^C(\mu_n), y^C(\mu_n))$ *(*$n = 1, 2, ...$*) is feasible and maximizes (11) subject to* $x_n \leq y_n, 0 \leq b_n$ *and* $0 \leq b_n + s_n$ *for all* $n = 1, 2, ....$

This result is imbedded in Lemma 1 in Appendix A. Although the proof of Lemma 1 applies here, we sketch the proof of Proposition 4 alone by confirming feasibility and unconstrained optimality. The latter step concerning $y$ is based on the following equation imbedded in Lemma 1 whose inverse is the right side of (20):

$$F(y - \mu) = \frac{\beta r - (1 - \beta - \rho)h - c}{\beta(r + h - c)}$$

Unconstrained optimality concerning $s$, i.e., (21), can be evaluated when $y_n = \mu_n + f$ for each $n$, regardless of whether $f = f_0$ or not. The proof of the following result is contained in the Appendix.

**Proposition 5.** *(a) If* $y = \mu + f$ *and* $s < 0$, *then* $L[(-s)^+, s, \mu + f, \mu]$ *is maximized by*

$$s = hf - (r + h)f_1 - r\mu \tag{22}$$

*(b) If* $y = \mu + f$ *and* $s \geq 0$, *then* $L[(-s)^+, s, \mu + f, \mu]$ *is maximized by*

$$s = hf - (r + h)f_2 - r\mu \qquad f_2 = F^{-1}\left[\frac{1 - \beta}{\beta a}\right] \tag{23}$$

*(c) If* $x_1 \leq \mu_1 + f$ *then* $y_n = \mu_n + f$ *is feasible for all* $n = 1, 2, ...$
*(d) If* $y_k = \mu_k + f$ *and* $s_k = hf - (r + h)f_1 - r\mu_k$ *for* $k = n - 1$ *and* $k = n$ *then*

$$\begin{aligned} v_n =& r(1 - \rho)\mu_n - \rho(r + h)f_1 - a(r + h)[(f - \varepsilon_{n-1})^+ + f_1 - f]^+ \\ &+ (r + h\rho)f - (r + h)(f - \varepsilon_{n-1})^+ \end{aligned} \tag{24}$$

In (20), $y^C(\mu)$ equals mean demand plus a fudge factor, $f_0$, that is constant with respect to firm size but depends on operational, financial and market factors. The operational factors are the holding cost rate $h$ and the unit production cost $c$, the financial factors are the discount factor $\beta$ and the short-term borrowing interest rate $\rho$, and the market factor is the unit price $r$ (and $F$). In addition to the obvious dependence on the firm size $\mu$, both $y^C(\mu)$ and $s^C(\mu)$ in (20) and (21) share the monotonicity properties of the myopic optimal solution in Li et al. (1997) (cf. Proposition 4.2 in Li et al. (1997)). In particular, $y^C(\mu)$ is nondecreasing with respect to $r$, $\beta$ (if $\rho \leq 1$), and $\rho$, and is nonincreasing with respect to $c$ (if $\rho \leq 1$) and $h$ (if $c/r \leq (1 - \rho)/(2 - \rho - \beta)$). Similarly, $s^C(\mu)$ is nondecreasing with respect to $a$, $\rho$, and $r$, and is nonincreasing with respect to $c$ and $h$.

In (22) and (23) the level of retained earnings diminishes at rate $r$ as the firm grows; i.e., larger firms borrow more. On the other hand, that level rises at rate $h$ (the holding cost rate) as the inventory buffer $f$ grows. The relevance to $s$ in Proposition 4 is that (19) implies that $s^C(\mu_n) < 0$

for each $n$. So (22) and $f = f_0$ imply $s = s^C(\mu)$. Later in the paper we discuss the property of (24) that the dividend grows with firm size at the rate of $r(1-\rho)$.

Proposition 4 is useful because many combinations of parameters $\beta, \rho, c, r, h, a$, and $F(\cdot)$ that are likely to arise in practice satisfy assumption (19). In addition to the restrictions on the scalar parameters $\beta, \rho, c, r, h$, and $a$ which are generally satisfied when the coefficients $r$ and $a$ are not particularly low and the borrowing interest rate $\rho$ is not excessively high in comparison to the discount rate $\beta$, (19) restricts the distribution function. For a given $F(\cdot)$, say normal with a mean of 100 units and a standard deviation of 30 units, (19) is satisfied for a wide range of values of $\beta, \rho, c, r, h$, and $a$. The normality assumption is tenable here because the coefficient of variation is less than one-third; so the probability of a negative value of $\varepsilon$ is very small. We found that the condition is most sensitive to the cost of borrowing $\rho$ and the default penalty function coefficient $a$. Values of either one that are high enough to violate (19) call for a lower value of $y$ and a higher value of $s$ than given in (20) and (21).

For example, let $r = \$10$ and suppose that the firm's gross margin in 50%; so $c = \$5$. Suppose that the annual holding cost rate is $h = \$1.25$ (25% of $c$), the discount rate $\beta = 0.8$, and the borrowing rate $\rho = 0.80$. Let $\varepsilon$ be normally distributed with mean 100 and standard deviation 30. If the default penalty rate is $a = \$10$, these values satisfy (19), (20), and (21) with $y^* = 107.34$, $y^C(\mu) = 105.67 + \mu$ and $s^C(\mu) = -236.43 - 10\mu$. The optimal short-term loan is $b = 236.43 + 10\mu$. If $a \in [0.25, 76]$, then (19) remains valid, $y^C(\mu)$ remains the same, and $s^C(\mu)$ ranges from $-992.90 - 10\mu$ to $-10\mu$. If $h$ were to increase from $h = \$1.25$ to $h = \$4$ (80% of the production cost $c$), then (19) would remain valid while $a \in [0.35, 19.25]$. The optimal supply level would become $y^C(\mu) = 89.33 + \mu$ and $s^C(\mu)$ would range from $-888.89 - 10\mu$ to $-10\mu$.

# 6 Firm Size and the Worth of Coordination

This section examines the influence of firm size on the extent to which it is worthwhile to coordinate production and financial decisions rather than leaving them decentralized. First, we consider the influence of firm size on optimal coordinated decisions. It seems intuitive that the following managed quantities should increase as the firm grows: inventory level, size of short-term loan, and dividend. However, some of the rates of increase may not be intuitive.

In this section we assume that the initial inventory is sufficiently low to permit $y_1 = \mu_1 + f_0$, i.e., $x_1 \leq \mu_1 + f_0$ and, therefore, that $y_n = \mu_n + f_0$ for all $n$, as stipulated by policy $\pi^C$, is feasible. We refer to $\pi^C$ as the "optimal coordinated" policy.

The following assertion is an immediate consequence of the roles of $\mu$ in (20) and (21) and of $\mu_k$ in (24), and of $b = (-s)^+$.

**Proposition 6.** *The optimal coordinated policy $\pi^C$ induces a physical goods base-stock level, short-term loan, and dividend that increase linearly with firm size. The respective growth rates are 1, $r$, and $r(1-\rho)$. The residual retained earnings decreases with firm size at rate $r$.*

The least intuitive part of this result is that the dividend growth rate is $r(1-\rho)$. In the demand model (5), $\mu_n$ is a "sure thing;" that is, the firm's revenue in period $n$ is at least $r\mu_n$ (because $\varepsilon_n \geq 0$). So the "sure" revenue $r\mu$ grows at rate $r$ as a function of $\mu$ (the same rate at which borrowing increases and retained earnings decreases). Policy $\pi^C$ directs a dividend-maximizing firm to borrow "against" any increase in the sure revenue stream and immediately to pass the *net* borrowing proceeds to shareholders in the form of a dividend. Since $\rho$ is the interest rate on short-term loans, the rate of change (with respect to $\mu$) of the net borrowing proceeds is $r(1-\rho)$.

This phenomenon occurs in practice as asset securitizations and asset-backed borrowing. These actions are frequently utilized by financially sophisticated firms to borrow at favorable rates against future revenue streams.

### Decentralized Operations and Financial Decisions

The optimal coordinated policy is more profitable than any decentralized policy because the model does not include the infrastructure costs of coordination. In practice, coordination would be worthwhile only if the difference were to exceed the infrastructure costs. This subsection specifies that difference and relates it to firm size. The preliminary issue is how to assess (10), the EPV (expected present value) of the dividends associated with a decentralized policy. The comparable EPV of the optimal coordinated policy is $\psi(x_1, \mu_1)$, namely the value function of the dynamic program at the initial state.

We assign an EPV to decentralized decisions by assuming that production decisions optimize the EPV of net profit, i.e., they are made without considering the consequences for borrowing, dividends, and bankruptcy. Therefore, the production decisions are made under the implicit assumption that the Miller-Modigliani theorem is applicable to the firm. The borrowing and dividend decisions are made subsequently; so financial management takes as given the cash flows induced by the production decisions.

As a result of these behavioral assumptions, operations maximizes the expected value of the following random variable:

$$B^o = \sum_{n=1}^{\infty} \beta^{n-1}[g(y_n, D_n) - cz_n]$$

This is the EPV of the revenues net of inventory- and production-related costs. The substitution of (5), (16), $z_n = y_n - x_n$, and $x_n = (y_{n-1} - D_{n-1})^+$, and a rearrangement of terms yields

$$B^o = cx_1 + \sum_{n=1}^{\infty} \beta^{n-1}[(r-c)y_n - (r+h-\beta c)(y_n - \mu_n - \epsilon_n)^+]$$

Since the expected value of the $n^{th}$ summand is a newsvendor objective, when $\mu_n = \mu$ it is maximized by

$$y = \mu + f_3 \qquad f_3 = F^{-1}\left[\frac{r-c}{r+h-\beta c}\right] \tag{25}$$

That is, if $x_1 \leq \mu_1 + f_3$, then the myopic decisions $y_n = \mu_n + f_3$ (for all $n = 1, 2, ...$) are feasible and (sub-)optimal. In the decentralized mode, we assume operations makes these decisions.

Next, finance selects short-term loans and dividends to maximize $E(B)$ given by (10) in which $y_n = \mu_n + f_3$ and $b_n = (-s_n)^+$ (for all $n = 1, 2, ...$). The arguments that lead to (11) yield the objective of maximizing

$$E \sum_{n=1}^{\infty} \beta^{n-1} L[(-s_n)^+, s_n, \mu_n + f_3, \mu_n] \tag{26}$$

Proposition 5 implies that the value of $s$ that maximizes $L[(-s)^+, s, \mu + f_3, \mu]$ is

$$s = hf_3 - r\mu - (r+h)f_1 \tag{27}$$

if $s < 0$. If $s \geq 0$, the solution is

$$s = hf_3 - r\mu - (r+h)f_2 \tag{28}$$

Let $\pi^D$ be a policy that employs (25), (27), and $b_n = (-s_n)^+$ with $\mu = \mu_n$ if $x_n \leq \mu_n + f_3$, and chooses $(b_n, s_n, y_n)$ arbitrarily but feasibly if $x_n > \mu_n + f_3$ (the mnemonic D denotes "decentralized"). Let $y^D(\mu)$ and $s^D(\mu)$ denote the values of $y$ and $s$ in (25) and (27), respectively, let $b^D(\mu) = (-s^D(\mu))^+$, and let $\Delta$ label the difference between EPVs of the dividends induced by $\pi^C$ and $\pi^D$. This is the difference between the maximal values of (11) and (26). The Appendix contains a proof of the following result.

**Proposition 7.** *$\Delta$ does not depend on the initial size of the firm, namely $\mu_1$.*

Proposition 7 implies that $\Delta/\mu_1 \to 0$ as $\mu_1 \to \infty$. So it is particularly worthwhile for small firms to coordinate production and financial decisions.

Next we show that the coordinated policy induces lower levels of inventory and retained earnings, and larger short-term loans than the decentralized policy. The Appendix contains the proof of the following result.

**Proposition 8.** *(a) $y^C(\mu) \leq y^D(\mu)$, $s^C(\mu) \leq s^D(\mu)$, and $b^C(\mu) \geq b^D(\mu)$.*

*(b) The rates of change with respect to $\mu$ of the physical goods base-stock level, retained earnings level, short-term loan, and dividend are the same under $\pi^C$ and $\pi^D$.*

# 7 Financial Distress

This section addresses two issues associated with financial distress. First, we derive the default probability that is induced by the optimal coordinated policy $\pi^C$. Second, we analyze the consequences if bankruptcy forces irrevocable dissolution of the firm; we call this "wipeout bankruptcy." Thus far in the model, bankruptcy has led to reorganization of the firm with attendant costs; we call this "reorganization bankruptcy."

### Probability of Bankruptcy and the Borrowing Rate of Interest $\rho$

Reorganization bankruptcy occurs at the beginning of period $n+1$ if the residual retained earnings is negative, i.e., $w_{n+1} < 0$. Since

$$w_{n+1} = s_n + g(y_n, D_n) = s_n + ry_n - (r+h)(y_n - \mu_n - \varepsilon_n)^+,$$

default occurs if $\varepsilon_n$ takes a value in the set $\{e : s + ry < (r+h)(y - \mu - e)^+\}$. Under the optimal coordinated policy, $y$ and $s$ are given by (20) and (21); so $y - \mu - e = f_0 - e$ and $s + ry = (r+h)(f_0 - f_1) > 0$ with the inequality due to assumption (19). Therefore, default occurs if $\varepsilon_n$ takes a value in

$$\{e : s + ry < (r+h)(f_0 - e)^+\} = \{e : e < f_0 - (s+ry)/(r+h\} = \{e : e < f_1\}$$

Let $q$ denote the probability of default in period $n$. Since $q = P\{e < f_1\}$ regardless of $\mu_n$, the default probability is the same every period and is invariant with respect to the size of the firm. So the duration of a sequence of default-free periods does not depend on the size of the firm at the beginning of the sequence and it has a geometric distribution (as in Li et al. (1997)).

**Proposition 9.** *Under the optimal coordinated policy, default occurs in a sequence of periods comprising a renewal process. The interval of time between successive defaults has a geometric distribution that does not depend on the firm's initial size.*

Comments

(i) This result implies that it is reasonable for the discount factor ($\beta$) and the borrowing interest rate ($\rho$) to be constants because the default risk is constant.
(ii) The intuition for this result lies in the additive structure of the autoregressive demand model.
(iii) Li et al. (1997) reach this conclusion for the special case of our demand model with $K = 0$.
(iv) If the reasoning that yields Proposition 9 is employed in the model with wipeout bankruptcy, it yields the conclusion that the lifetime of the firm has a geometric distribution that does not depend on the firm's initial size.

### The Option to Declare Bankruptcy in Wipeout Bankruptcy

Bankruptcy in reality is a complex legal process that we have thus far simplified by charging a default penalty when retained earnings is negative, i.e., $w_{n+1} < 0$ leads to the costs of restructuring debt. In this subsection, we consider the alternative scenario of wipeout bankruptcy. Without the opportunity to restructure debt, the firm has the option to declare bankruptcy either by "looting the treasury" (letting $v_n = w_n$ and $z_n = b_n = 0$) or by "walking away," i.e., surrendering its assets if the demand in period $n$, $D_n$, fails to generate sufficient revenue thus causing $w_{n+1} = s_n + g(y_n, D_n) < 0$. The latter is particularly important when the firm relies on external borrowing because one of the model's implicit assumptions is that the ownership structure of the firm is consistent with a limited liability corporation. That gives the firm the option to return its assets to the lenders for the loans's notional value at maturity.

Let $q^c(s, y, \mu)$ denote the probability that bankruptcy does not occur in period $n+1$ if $s_n = s, y_n = y$, and $\mu_n = \mu$. An extention of the argument in Li et al. (1997) is applicable here and shows that dynamic program (13) remains valid with wipeout bankruptcy when (13b) is replaced with

$$\begin{aligned} J(b, s, y, \mu) &= L_w(b, s, y, \mu) + \beta q^c(s, y, \mu) E\{\psi[(y-\mu-\varepsilon)^+, \theta(\mu+\varepsilon)]\} \\ L_w(b, s, y, \mu) &= q^c(s, y, \mu) E[\beta g(y, \mu+\varepsilon) - (1-\beta)s \\ &\quad - cy + \beta c E[(y-\mu-\varepsilon)^+] - \rho b] - (s + cy + \rho b)(1 - q^c(s, y)), \end{aligned}$$

Observe that $L_w(b+\mu, s+\mu, y+\mu, \mu)$ is constant with respect to $\mu$; so $\Lambda_w(b, s, y) = L_w(b+\mu, s+\mu, y+\mu, \mu)$ is well-defined as is the following dynamic program with $\Psi(x) = \psi(x-\mu, \mu)$:

$$\begin{aligned} \Psi(x) &= \sup_{b,s,y}\{M(b, s, y) : y \geq x, b \geq 0, b+s \geq 0\} \\ M(b, s, y) &= \Lambda_w(b, s, y) + \beta q^c(s, y) E(\Psi[(y-\varepsilon)^+]) \end{aligned}$$

This dynamic program is formally the same one that is analyzed in (Li et al. (1997), §5). Therefore, there are optimal base-stock levels for physical goods inventory and retained earnings, and the lifetime of the firm is geometrically distributed with a parameter that depends only on the optimal base-stock levels (and not on firm size).

The key insights from this section are (a) the default probability is constant with respect to firm size, (b) changes in the bankruptcy mechanism have only a scaling effect on the optimal base-stock levels, and (c) the model with wipeout bankruptcy shares key properties of the model with reorganization bankruptcy and in some ways is easier to analyze.

# 8 Conclusions and Generalizations

We address the coordination of production and financial decisions in a firm that maximizes the expected present value of dividends and that encounters demand that evolves according to an autoregressive process. The firm can borrow funds on a short-term basis, which in turn exposes it to the risk of bankruptcy. We use the deterministic part of demand as the proxy for firm size, $\mu$.

The contributions include the complete characterization of an optimal policy that is myopic and entails base-stock levels for physical goods inventory and retained earnings. The optimal borrowing strategy follows the well-known pecking-order hypothesis. The base-stock level for inventory equals $\mu$ plus a factor, and the one for retained earnings equals $-r\mu$ plus a factor ($r$ is the exogenous price for the firm's product). Neither factor depends on $\mu$. The optimal policy yields dividends that grow with firm size at the rate of $r(1-\rho)$ ($\rho$ is the interest rate for borrowing short-term).

Since the infrastructure costs of coordination are not included in the model, it is necessarily better to coordinate production and financial decisions than to decentralize them. We compare the effects of optimal coordinated and decentralized decisions and find repeatedly that the relative improvement of coordination is greater for small firms rather than for large.

It is important to examine the dependence of the conclusions on the assumptions. For example, bankruptcy is a complex financial and legal process, and in most of the paper we assume that the firm's inability to repay a loan leads to restructuring its debt with associated costs. In §7 we explain why the replacement of the restructuring assumption with irrevocable dissolution of the firm would preserve key results.

Following preliminary results (Propositions 1, 2, and 3) we assume particular piece-wise linear expressions for the default penalty and the sales revenue net of inventory-related costs. Most of the qualitative results would be preserved with a general default penalty function that is nonincreasing and convex, and a general net sales revenue function that is concave. However, simple explicit formulas would be lost.

In §3 we introduce a general autoregressive demand process (4), but we employ a first-order process thereafter for expository simplicity (i.e., $K = 1$ after (4)). The use of a higher-order process would preserve most of the paper's qualitative results, but simple explicit formulas would be lost.

In the model the firm pays a constant rate of interest for short-term loans, but the qualitative results would be preserved if, instead, the interest rate were a convex nondecreasing function of the probability of default. However, §7 shows that this generality is unnecessary because the probability of default remains constant. Similarly, the model uses a constant single-period discount factor, and converting this constant to a function is unnecessary.

# Appendix

# A The Complete Solution

## A.1 Kharush-Kuhn-Tucker Conditions

In §5, $(b, s, y) = (b^C(\mu), s^C(\mu), y^C(\mu))$ maximizes $L(b, s, y, \mu)$ in (17) subject to $b \geq 0$ and $b + s \geq 0$. The KKT conditions for this problem are as follows:

$$\frac{\partial L(b, s, y, \mu)}{\partial s} + \lambda_1 = 0 \tag{29a}$$

$$\frac{\partial L(b, s, y, \mu)}{\partial y} = 0 \tag{29b}$$

$$\frac{\partial L(b, s, y, \mu)}{\partial b} + \lambda_1 + \lambda_2 = -\rho + \lambda_1 + \lambda_2 = 0 \tag{29c}$$

$$\lambda_1(b + s) = 0 \tag{29d}$$

$$\lambda_2 b = 0 \tag{29e}$$

Equations (29a) and (29b) are given by

$$\frac{\partial L(b, s, y, \mu)}{\partial s} = -(1 - \beta) + \begin{cases} \beta a F\left(\frac{hy - s - \mu(r+h)}{(r+h)}\right), & \text{if } s + ry > 0; \\ \beta a, & \text{if } s + ry < 0. \end{cases}$$

and

$$\begin{aligned} \frac{\partial L(b, s, y, \mu)}{\partial y} &= -c + \beta r - \beta(r + h - c)F(y - \mu) \\ &+ \begin{cases} -\beta a h F\left(\frac{hy - s - \mu(r+h)}{(r+h)}\right), & \text{if } s + ry > 0; \\ \beta a[r - (r + h)F(y - \mu)], & \text{if } s + ry < 0. \end{cases} \end{aligned}$$

because the expected value of the penalty function $p(\cdot)$ given by (15) is:

$$\begin{aligned} &E\{a[s + ry - (r + h)(y - \mu - \varepsilon)^+]^-\} \\ &= \begin{cases} -a \int_0^{[hy - s - \mu(r+h)]/(r+h)} [hy - s - \mu(r + h) \\ \quad -(r + h)x] f(x)dx, & \text{if } s + ry > 0; \\ -a \int_0^{y-\mu} (r + h)(y - \mu - x) f(x)dx, & \text{if } s + ry = 0\ ; \\ a\{s + ry - (r + h) \int_0^{y-\mu} (y - \mu - x) f(x)dx\}, & \text{if } s + ry < 0. \end{cases} \end{aligned}$$

## A.2 Complete Solution

**Lemma 1.** *If $(1-\beta)c \leq \beta r$, an optimal value of $(b, s, y)$ can be determined as follows. Let $y_1$ solve:*

$$-c + \beta r - \beta(r+h-c)F(y_1 - \mu) - \beta a h F\left(\frac{hy_1 - \mu(r+h)}{r+h}\right) = 0 \tag{30}$$

1. *If*

$$\frac{1-\beta}{\beta a} < F\left(\frac{hy_1 - \mu(r+h)}{r+h}\right)$$

   *then $y^C(\mu) = y_0$, $s^C(\mu) = s_0$ and $b^C(\mu) = 0$ where*

$$F(y_0 - \mu) = \frac{\beta r - (1-\beta)h - c}{\beta(r+h-c)}$$

$$F\left(\frac{hy_0 - \mu(r+h) - s_0}{r+h}\right) = \frac{1-\beta}{\beta a}$$

2. *If*

$$\frac{1-\beta-\rho}{\beta a} \leq F\left(\frac{hy_1 - \mu(r+h)}{r+h}\right) \leq \frac{1-\beta}{\beta a}$$

   *then the solution is $y^C(\mu) = y_1$ and $s^C(\mu) = b^C(\mu) = 0$.*

3. *If*

$$F\left(\frac{hy_1 - \mu(r+h)}{(r+h)}\right) < \frac{1-\beta-\rho}{\beta a} \tag{31}$$

   *then:*

   (a) *If*

$$\frac{1-\beta-\rho}{\beta a} < \frac{\beta r - (1-\beta)h - c + h\rho}{\beta(r+h-c)} \tag{32}$$

   *then $y^C(\mu) = y_2$, $s^C(\mu) = s_2$ and $b^C(\mu) = -s$ where*

$$F(y_2 - \mu) = \frac{\beta r - (1-\beta)h - c + h\rho}{\beta(r+h-c)} \tag{33}$$

$$F\left(\frac{hy_2 - s_2 - \mu(r+h)}{r+h}\right) = \frac{1-\beta-\rho}{\beta a} \tag{34}$$

   (b) *If*

$$\frac{\beta r - (1-\beta)h - c + h\rho}{\beta(r+h-c)} \leq \frac{1-\beta-\rho}{\beta a} < 1 \tag{35}$$

   *then $y^C(\mu) = y_3$, $s^C(\mu) = -ry_3$ and $b^C(\mu) = ry_3$ where*

$$F(y_3 - \mu) = \frac{(1-\rho)r - c}{\beta a(r+h) + \beta(r+h-c)}$$

*(c) If*

$$1 \leq \frac{1-\beta-\rho}{\beta a}$$

*then* $y^C(\mu) = y_4$, $s^C(\mu) = -\infty$ *and* $b^C(\mu) = \infty$ *where*

$$F(y_4 - \mu) = \frac{\beta r(1+a) - c}{\beta a(r+h) + \beta(r+h-c)}$$

### A.3 Additional Comments on Sufficient Conditions

The discussion in the remainder of this paragraph appeals to Lemma 1. Inequalities (18) and (19) are special cases of (30), (31) and (32) with $\mu = 0$. Lemma 2 establishes that (19) implies that (31) is valid for all $\mu \geq 0$. Therefore, $(b^C(\mu), s^C(\mu), y^C(\mu)) = (b_2, s_2, y_2)$ for all $\mu \geq 0$.

Moreover, if (32) is valid (this is the same as the right-hand inequality in (19)) and

$$\frac{1-\beta-\rho}{\beta a} < F\left(\frac{hy_1(\mu) - \mu(r+h)}{r+h}\right) \text{ at } \mu = 0$$

where $y_1(\mu)$ solves (30), then Lemma 3 in §E implies that (31) eventually holds as $\mu$ increases. The implication of this is that even if the left-hand inequality of (19) is invalid, as $\mu$ increases, all results in the main body of the paper that require (19) become valid.

Note that this argument assumes that (32) is valid. This condition prevents the optimal solution being $(b^C(\mu), s^C(\mu), y^C(\mu)) = (b_3, s_3, y_3)$ or $(b_4, s_4, y_4)$, which is not of great practical interest. Inspection of Lemma 1 shows that the former requires $b^C(\mu) = ry^C(\mu)$ implying that default will be triggered by any amount of inventory; and the latter requires $b^C(\mu) = \infty$ implying that the firm will default with probability one.

# B Proof of Proposition 5.

*Proof.* Confirm (a) and (b) by using (17) in $L[(-s)^+, s, \mu + f, \mu]$ and applying Leibnitz's rule to the maximization problems when $s < 0$ and $s \geq 0$. For (c), if $x_n \leq \mu_n + f$ and $y_n = \mu_n + f$ then $x_{n+1} = (y_n - D_n)^+ = [(\mu_n + f) - (\mu_n + \varepsilon_n)]^+ = (f - \varepsilon_n)^+ \leq f \leq \mu_{n+1} + f = y^C(\mu_{n+1})$.

For (d), in the definition of $w_n$ in (6b), substitute $z_n = y_n - x_n$, $x_n = (y_{n-1} - D_{n-1})^+$, $w_n = g(y_n, D_n) + s_n$, $g(y_{n-1}, D_{n-1}) = ry_{n-1} - (r+h)(y_{n-1} - D_{n-1})^+$, $y_n = \mu_n + f$, $y_{n-1} = \mu_{n-1} + f$, and $D_{n-1} = \mu_{n-1} + \varepsilon_{n-1}$, and use $s_{n-1} - s_n = r(\mu_n - \mu_{n-1})$ from (22) to obtain

$$\begin{aligned}
v_n &= w_n - s_n - p(w_n) - cz_n - \rho b_n \\
&= g(y_{n-1}, D_{n-1}) + s_{n-1} - s_n - p[g(y_{n-1}, D_{n-1}) + s_{n-1}] \\
&- cy_n + c(y_{n-1} - D_{n-1})^+ - \rho b_n \\
&= rf - (r+h)(f - \varepsilon_{n-1})^+ + r\mu_n - a(r+h)[(f - \varepsilon_{n-1})^+ + f_1 - f]^+ \\
&- r\rho\mu_n + h\rho f - \rho(r+h)f_1
\end{aligned}$$

which yields (24). $\square$

# C Proof of Proposition 7

*Proof.* The pairs $\big((20), (21)\big)$ and $\big((25), (27)\big)$ are the same except that $f_0$ in the former is replaced by $f_3$ in the latter. Since (19) ensures $s_n < 0$ in both pairs (for all $n$), a rearrangement of terms yields the following expression that does not depend on $\mu_1$:

$$\begin{aligned}
\Delta &= E\sum_{n=1}^{\infty} \beta^{n-1}\big[L\big([-hf_3 + r\mu + (r+h)f_1]^+, hf_3 - r\mu - (r+h)f_1, \mu_n + f_3, \mu_n\big) \\
&- L\big([-hf_0 + r\mu + (r+h)f_1]^+, hf_0 - r\mu - (r+h)f_1, \mu_n + f_0, \mu_n\big)\big] \\
&= (1-\beta)^{-1}[\beta(r+h) + \rho h](f_0 - f_2) - \beta(r+h-c)E\big([(f_0 - \varepsilon)^+ - (f_2 - \varepsilon)^+] \\
&- \beta a(r+h)E\big([(f_0 - \varepsilon)^+ - f_0 + f_1]^+ - [(f_2 - \varepsilon)^+ - f_2 + f_1]^+\big)
\end{aligned}$$

$\square$

# D Proof of Proposition 8

*Proof.* (a) From (20) and (25), $y^C(\mu) \le y^D(\mu)$ corresponds to $f_0 \le f_3$ which is equivalent to $0 \le \delta$ where

$$\delta = (r-c)\beta(r+h-c) - (r+h-\beta c)[\beta r - (1-\beta-\rho)h - c]$$

Rearranging terms,

$$\begin{aligned}
\delta &= (r-c)\beta(r+h-c) - (r+h-\beta c)(\beta r - c) + h(r+h-\beta c)(1-\beta-\rho) \\
&\ge (r-c)\beta(r+h-c) - (r+h-\beta c)(\beta r - c) + h(r+h-c)(1-\beta-\rho) \\
&= (1-\beta-\rho)h(r+h-c) + c(1-\beta)[r(1-\beta)+h] \ge 0
\end{aligned}$$

So $y^C(\mu) \le y^D(\mu)$. The inequality $f_0 \le f_3$, (21), and (27) imply $s^C(\mu) \le s^D(\mu)$. So $b^C(\mu) \ge b(\mu)$.
(b) Proposition 5. $\square$

# E Miscellaneous Technical Results

**Lemma 2.** *If (32) is valid, $y_1(\mu)$ solves (30) with $\mu = 0$, and (31) is valid with $y_1(\mu)$ at $\mu = 0$, i.e., if*

$$F\left(\frac{hy_1(0)}{r+h}\right) < \frac{1-\beta-\rho}{\beta a}$$

*then (31) is valid for all $\mu > 0$.*

*Proof.* Let $y_1(\mu)$ solve (30) for some $\mu > 0$. Since $F(\cdot)$ is nondecreasing, it is sufficient to show

$$\frac{hy_1(0)}{r+h} \geq \frac{hy_1(\mu) - \mu(r+h)}{r+h}$$

By contradiction, suppose

$$\frac{hy_1(0)}{r+h} < \frac{hy_1(\mu) - \mu(r+h)}{r+h} \tag{36}$$

So

$$y_1(0) < y_1(\mu) - \frac{\mu(r+h)}{h}$$

Assuming $1 < (r+h)/h$ and $y_1(0) \geq 0$,

$$y_1(\mu) - \mu > y_1(\mu) - \frac{\mu(r+h)}{h} \qquad \text{and} \qquad y_1(\mu) - \mu > y_1(0) \tag{37}$$

However, (36) and (37) imply

$$\begin{aligned} 0 &= -(1-\beta)c + \beta r - \beta(r+h)F(y_1(0)) - \beta a h F\left(\frac{hy_1(0)}{(r+h)}\right) \\ &\geq -(1-\beta)c + \beta r - \beta(r+h)F(y_1(\mu)-\mu) - \beta a h F\left(\frac{hy_1(\mu) - \mu(r+h)}{(r+h)}\right) \end{aligned} \tag{38}$$

(with strict inequality if $F(\cdot)$ is strictly increasing at either argument of $F(\cdot)$ on the right side of (38)) contradicting the fact that $y_1(\mu)$ solves (30) for some $\mu > 0$. $\square$

**Lemma 3.** *If (32) is valid, $F(\cdot)$ is strictly increasing at $[hy_1(\mu) - \mu(r+h)]/(r+h)$, $y = y_1(\mu)$ is a solution of (30) at $\mu = 0$, and*

$$F\left(\frac{hy_1(\mu) - \mu(r+h)}{r+h}\right) > \frac{1-\beta-\rho}{\beta a} \text{ at } \mu = 0$$

*then there exists $\mu^* > 0$ such that*

$$F\left(\frac{hy_1(\mu) - \mu(r+h)}{r+h}\right) < \frac{1-\beta-\rho}{\beta a}, \ \forall \mu \geq \mu^*$$

*Proof.* Let $\mu_1 = 0$ in Lemma 4 and use the fact that $F(\cdot)$ is strictly increasing. $\square$

**Lemma 4.** *If (32) is valid,* $F(\cdot)$ *is strictly increasing either at* $[hy_1(\mu_2) - \mu_2(r+h)]/(r+h)$ *or at* $(y_1(\mu_2) - \mu_2)$*, and* $y_1(\mu_1)$ *and* $y_1(\mu_2)$ *are solutions of (30) where* $0 \leq \mu_1 < \mu_2$*, then*

$$\frac{hy_1(\mu_1) - \mu_1(r+h)}{r+h} > \frac{hy_1(\mu_2) - \mu_2(r+h)}{r+h}$$

*Proof.* By contradiction, if

$$\frac{hy_1(\mu_1) - \mu_1(r+h)}{r+h} \leq \frac{hy_1(\mu_2) - \mu_2(r+h)}{r+h} \tag{39}$$

then (39) implies

$$h[y_1(\mu_1) - \mu_1] \quad \leq \quad h[y_1(\mu_2) - \mu_2] - r(\mu_2 - \mu_1) < h[y_1(\mu_2) - \mu_2]$$

or

$$y_1(\mu_2) - \mu_2 < y_1(\mu_2) - \mu_2 \tag{40}$$

But (39) and (40) imply that both $y_1(\mu_1)$ and $y_1(\mu_2)$ cannot solve (30). $\square$

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