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Capital Structure and Inventory Management

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Abstract

We examine the impact of a firm's capital structure on its short-term decisions concerning inventories, dividends, and liquidity. The model is a single-stage inventory system with the criterion of maximizing the value of the firm, namely the expected present value of dividends. In a linear-cost case, the model is a dynamic newsvendor, and there is an optimal base-stock inventory policy which does not depend on the firm's capital structure. However, the optimal dividend and liquidity policy depends on the inventory decisions. Without making linearity assumptions, the optimal capital structure depends on all parameters. However, sufficient conditions for the optimum to be all-debt or all-equity depend only on financial parameters. We also show that a base-stock policy, optimal or not, has the "smoothing effect" that the variance of dividends is less than the variance of profits.

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1 Operations and Capital Structure

Many firms now have enterprise-wide information systems, so it is technically feasible to coordinate decisions in separate functions. Feasibility raises several questions. First, are the incremental benefits of multi-function coordination sufficiently greater than the costs of coordination? Second, how would a firm behave if it were coordinating several functions? Third, how would customers and markets (and society at large) be affected by coordination? Here, we provide some answers to the second question for the functions of operations and finance. Also, illustrative calculations suggest that the benefits of coordination can be substantial.

For more than 40 years it has been known (Miller and Modigliani (1961)) that a firm's capital structure does not not affect its market value *if* the firm operates in capital markets that are perfect and complete. This fact is a rationale to separate production decisions from financial decisions. However, many firms are capital constrained, and this is particularly true if they are young and entrepreneurial. Such firms do *not* operate in perfect complete capital markets, and they may benefit from a coordinated production and financial policy. What are good coordinated policies? This paper examines (i) the normative impacts of a firm's capital structure on its inventory, dividend, and short-term borrowing decisions, (ii) the interactions among these decisions, and (iii) the implications for capital structure considering taxes and bankruptcy costs.

The corporate tax advantage of debt can be nullified by direct and indirect bankruptcy costs and the risk of bankruptcy. Direct costs include legal and administrative costs of liquidation or reorganization and indirect costs include damaged relationships with customers and suppliers, and "fire sale" liquidation of the firm's assets below their market value. The consequent research literature on optimal capital structure either trades off tax advantages versus bankruptcy (Kraus and Litzenberger (1973), J.H. Scott (1977), Brennan and Schwartz (1978), Kim (1978), Turnbull (1979), and DeAngelo and Masulis (1980)) or employs an agency theory framework (Myers (1977), Smith and Warner (1979)). We use the former framework and ignore agency issues.

If a firm operates in factor and product markets with uncertainties, then it risks being unable to meet its obligations to government (taxes) and private creditors (employees, suppliers, bond-holders, and banks). A higher debt-equity ratio corresponds to larger liabilities (higher coupon payments) thus a higher risk of bankruptcy. How should a firm manage this risk? How should it adjust its short-term financial decisions such as dividends and short-term borrowing? How should it adjust its production quantities? An empirical study (Phillips (1995)) finds that output in certain industries is negatively associated with the average industry debt-equity ratio. However, we are not aware of theoretical work on this question.

Until a decade ago financial considerations were conspicuously absent in the extensive literature on models of inventory and production processes (Graves, Kan and Zipkin (1993)). However, recent research on the coordination of operational and financial decisions (Archibald, Thomas, Betts and Johnston (2002), Buzacott and Zhang (2004), Xu and Birge (2004b), Xu and Birge (2004c), Xu and Birge (2004a), Li, Shubik and Sobel (1997), Sobel and J.Zhang (2003), and Babich and Sobel (2004)) does not address the effects of a firm's capital structure on its operational decisions.

Our basic premise is that capital structure affects operational performance and risk. That is, different capital structures invite different operational decisions. Since financial performance depends on operational performance, a capital structure should be selected with an awareness of the induced operational consequences. Our primary contribution is a demonstration that the interaction of capital structure and operational performance is amenable to analysis. We pose general issues in a discrete-time infinite-horizon model of a firm which makes an initial strategic decision on the amount of bonds to issue, and this decision results in periodic coupon payments. The firm faces product market risks (stochastic demand) and financial risks (bankruptcy), and the objective is to maximize the firm's market value via periodic decisions on product replenishment, dividends, and short-term borrowing. The market value is the expected present value of the time stream of dividends.

We formulate the model in §2, and in §3 we examine the effect of capital structure on short-term inventory and financial decisions. §4 identifies the mixture of long-term debt and equity which maximizes the value of the firm, and §5 contrasts the variances of dividends and profits. §6 briefly discusses the extent to which the results in earlier sections would be affected if the model encompassed multiple products or the treatment of bankruptcy were altered. §7 has concluding remarks.

A base-stock inventory replenishment policy has long been known to minimize a dynamic newsvendor's expected cost (Bellman, Glicksberg and Gross (1955)). A linearity assumption In §3 implies that a base-stock policy for inventory and residual retained earnings maximizes a firm's market value (Property 1). The optimal base-stock inventory level is invariant with respect to the amount of long-term debt, but the optimal level of residual retained earnings depends on the amount of long-term debt. The model with the linearity assumption can be viewed as a dynamic newsvendor who maximizes the firm's market value. Reverting to the general model in §4, we show that the use of a base-stock policy (optimal or not) implies that the optimal capital structure depends neither on short-term interest nor inventory replenishment cost (Property 2). Indeed, sufficient conditions for an optimal capital structure to be all-debt or all-equity depend only on financial parameters (Proposition 2). Continuing with the general model in §5, we show that a base-stock policy (optimal or not) causes the variance of dividends to be less than the variance of profits. (Proposition 3).

2 Model

We assume that the firm has initial levels of debt and equity and, thereafter, these levels remain constant. Let η be the amount of equity. Specifically, the bonds have an infinite maturity date and the proceeds are mQ which entails a periodic coupon payment of Q. Let w_n $(n = 1, 2, \cdots)$ be the firm's internal capital at the beginning of period n, so w_1 is the firm's initial working capital. In order to reflect the tax advantage attraction of debt-financing, let $1 - \tau$ be the firm's marginal income tax rate.

At the beginning of each period n $(n = 1, 2, \cdots)$, the firm knows w_n and the size of its physical goods inventory, denoted x_n, and makes three decisions: b_n, the amount of a short-term loan; z_n, the physical goods replenishment quantity; and v_n, the amount of dividends to issue. The repayment of b_n is due at the end of period n. The constraints on the decision variables are $b_n \geq 0$ and $z_n \geq 0$, but dividends are *not* constrained $v_n \geq 0$ for the following reason. Capital subscriptions occur frequently in entrepreneurial firms and if $v_n < 0$ we interpret $|v_n|$ as a capital subscription. Later in the section we comment on the effects of imposing the constraint $v_n \geq 0$.

Although a broad array of inventory replenishment models would be consistent with the model, for specificity and simplicity we assume that ordered goods are provided by a single-stage source without delay, excess demand is backlogged, and successive periods' demands are independent and identically distributed nonnegative random variables D_1, D_2, $\cdots$. Let $F(\cdot)$ denote the distribution function of D_1 and let D be a random variable with distribution function $F(\cdot)$. The model can be generalized in numerous ways including positive lead times, excess demand being lost, a multi-stage source for the ordered goods, and autocorrelated demands. Under the stated assumptions,

$$y_n = x_n + z_n \tag{1}$$

is the total amount of goods available to satisfy demand in period n. The constraint $z_n \geq 0$ corresponds to $y_n \geq x_n$.

We model the interest on the short-term loan b_n in period n as a random variable $\lambda_n(b_n, y_n)$ whose distribution depends on the amount borrowed and on the total supply of goods. This representation can reflect the dependence of the interest rate on the firm's current risk of bankruptcy, and it includes borrowing limits contingent on the firm's current condition. We assume that $\lambda_1(b, y), \lambda_2(b, y), ...$ are independent and identically distributed random variables (for each pair (b, y) with $b \geq 0$).

Bankruptcy laws span diverse consequences of insolvency and near-insolvency (Senbet and Seward (1995)). The model here embodies *reorganization* bankruptcy which refers to a costly restructuring and continuation of operations. The effects of *wipeout* bankruptcy, which refers to a permanent shut-down of operations, are briefly discussed in §6. The firm is declared insolvent at the end of a period if it has insufficient funds to pay the bond coupon Q and repay the short-term loan (if one was made at the beginning of the period). Insolvency at the end of period n leads to reorganization of the firm accompanied in period $n+1$ by the resumption of operations after payment of a bankruptcy penalty $p(w_{n+1})$. The costs of reorganization bankruptcy are both direct, such as fees paid to lawyers and accountants, and indirect, such as lost sales and damaged supplier and customer relationships.

We represent the sales revenue net of inventory costs in period n as a function $g(y_n, D_n)$ of total supply and demand. Let $L(z_n)$ be the cost incurred in period n to replenish the quantity $z_n = y_n - x_n$. We assume that the firm is subject to a liquidity constraint that obliges it to fund its expenditures early in the period: $w_n + b_n \geq v_n + \tau[L(z_n) + \lambda_n(b_n, y_n) + p(w_n)]$. The left side is the sum of retained earnings and the short-term loan. The right side is the sum of dividends and, net of tax credits, inventory replenishment cost, short-term interest, and bankruptcy penalty.

It is convenient to define

$$s_n \quad = \quad w_n - v_n - \tau[L(z_n) + \lambda_n(b_n, y_n) + p(w_n)] \tag{2}$$

which is the residual internal capital after making expenditures early in the period. Therefore, the total working capital available in period n is $w_n - v_n + b_n - \tau[\lambda_n(b_n, y_n) + p(w_n)]$. This interpretation of (2) is consistent with accrual accounting. Thus, the liquidity constraint is

$$b_n + s_n \geq 0 \tag{3}$$

and other constraints on the decision variables are

$$b_n \geq 0 \quad and \quad y_n \geq x_n \tag{4}$$

We note that a limitation on short-term borrowing capacity can be represented either as an upper bound in (4) or as a prohibitively high interest charge in λ_n.

Under quite general conditions, it can be shown (as in Li et al. (1997)) that an optimal policy specifies $b_n = (-s_n)^+$ in each period n, so $b_n > 0$ implies $s_n = -b_n$. Therefore, the distribution of the interest random variable $\lambda_n(b_n, y_n)$ depends on the residual internal capital.

We model the single-period discount factor in period n as a random variable $\beta_n(s_n, y_n)$ whose distribution depends on the residual internal capital and the total supply of goods.

That is, the firm's opportunity cost depends on its residual internal capital, its supply of goods, and phenomena beyond the firm's control. We assume that $\beta_1(s,y), \beta_2(s,y), ...$ are independent and identically distributed random variables (for each pair (s,y)), and that $(D_1, D_2, ...), (\lambda_1(b,y), \lambda_2(b,y), ...)$, and $(\beta_1(s,y), \beta_2(s,y), ...)$ are mutually independent sequences (for each (b,s,y) with $b \geq 0$).

The dynamics reflect the backlogging of excess demand and the balancing of cash flows:

$$x_{n+1} = y_n - D_n \tag{5}$$

$$w_{n+1} = s_n + \tau[g(y_n, D_n) - Q] \tag{6}$$

The value of the firm is the maximal value of $E(B)$ where the random variable B denotes the present value of dividends. Using (2), (5), and (6) (and the convention $\Pi_1^0 \equiv 1$) yields

$$\begin{aligned} B &= \sum_{n=1}^{\infty} [\Pi_{j=1}^{n-1} \beta_j(s_j, y_j)] v_n \\ &= \sum_{n=1}^{\infty} [\Pi_{j=1}^{n-1} \beta_j(s_j, y_j)] \{w_n - s_n - \tau[L(z_n) + \lambda_n(b_n, y_n) + p(w_n)]\} \\ &= \sum_{n=1}^{\infty} \sigma_n^* \mathcal{K}_n(b_n, s_n, x_n, y_n; Q) + w_1 - p(w_1) \end{aligned} \tag{7}$$

with the definitions

$$\sigma_n^* = [\Pi_{j=1}^{n-1} \beta_j(s_j, y_j)] \tag{8}$$

$$\begin{aligned} \mathcal{K}_n(b,s,x,y;Q) = &-[1 - \beta_n(s,y)]s + \tau\beta_n(s,y)[g(y,D_n) - p\{s + \tau g(y,D_n) - \tau Q\} - Q] \\ &- \tau[L(y-x) - \lambda_n(b,y)] \end{aligned}$$

where it is noteworthy that $\mathcal{K}_n$ does not depend on w_n, and $\mathcal{K}_1(b,s,x,y,Q)$, $\mathcal{K}_2(b,s,x,y,Q)$, ... are independent and identically distributed random variables (for each b, s, x, y, and Q).

Therefore,

$$\begin{aligned} E(B) &= E\{\sum_{n=1}^{\infty} [\Pi_{j=1}^{n-1} \beta_j(s_j, y_j)] v_n\} \\ &= E[\sum_{n=1}^{\infty} \sigma_n K(b_n, s_n, x_n, y_n; Q)] + w_1 - p(w_1) \end{aligned} \tag{9}$$

with the definitions

$$\sigma_n = E(\sigma_n^*) \qquad K(b,s,x,y,Q) = E[\mathcal{K}_1(b,s,x,y,Q)] \tag{10}$$

3 Optimal Policies and Impacts of Long-Term Debt

In this section we characterize an optimal policy and, by fixing the size of the coupon payment, Q, we study the impacts of Q on production, short-term borrowing, and liquidity (hence on dividends).

Given the state vector (x_n, w_n), the decision variables s_n, b_n, and y_n correspond to the decision variables v_n, b_n, and z_n. So define a *policy* to be a non-anticipative rule for making decisions such that, for each n, (s_n, b_n, y_n) satisfies (3) and (4) and is a function of the elapsed history. An *optimal policy* maximizes the value of the firm, namely $E(B|x_1 = x, w_1 = w)$, for each initial state (x, w). The goal is to characterize an optimal policy and to relate it to the long-term debt level.

The model here is a generalization of that in Li et al. (1997) but its optimal policy retains three simplifying properties. First, given an initial choice of Q and w_1, it follows from (9) that an optimal policy does *not* depend on $\{w_n\}$. That is, an optimal policy corresponds to the following dynamic program whose state consists of the scalar inventory level x rather than the vector (x, w) where w is the generic level of internal capital at the beginning of a period:

$$\psi(x) = max\{J(b, s, x, y; Q) : y \geq x, b \geq 0, b + s \geq 0\}$$
$$J(b, s, x, y; Q) = K(b, s, x, y; Q) + E[\beta_1(s, y)\psi(y - D)] \tag{11}$$

If a firm issues bonds with proceeds mQ, issues stock with proceeds $w_1 - mQ$, and has an initial inventory of x_1 then it follows from (9) that the market value is $\psi(x_1; Q) + w_1 - p(w_1)$.

The second property of an optimal policy that occurs too in Li et al. (1997) is $b_n = (-s_n)^+$. That is, it follows from (11) that if $E[\lambda_1(\cdot, y)]$ is nondecreasing (given any y), then $b_n = (-s_n)^+$, $n = 1, 2, ...$ is optimal. The assumption that total interest rises with the amount borrowed is innocuous, and the conclusion is intuitive. Don't borrow unless you are forced to do so, and then borrow as little as possible. This property is consistent with pecking order theory that asserts that firms should use internally generated funds before resorting to short-term borrowing, long-term borrowing and issuing stock.

The third property is that a linear cost of replenishing inventory leads to a myopic optimum. That is, let $L(z) = cz$ where c is the unit cost of replenishment, substitute $L(z_n) = cz_n$ in (7), and use $z_n = y_n - x_n$ and $x_n = y_{n-1} - D_{n-1}$ in the same series of steps that leads from (7) to (9). As a result,

$$
\begin{aligned}
E(B) &= E\{\sum_{n=1}^{\infty}[\Pi_{j=1}^{n-1}\beta_j(s_j, y_j)]v_n\} \\
&= E[\sum_{n=1}^{\infty}[\Pi_{j=1}^{n-1}\beta_j(s_j, y_j)]K(b_n, s_n, y_n; Q)] + w_1 - p(w_1) + \tau c[x_1 - \sum_{n=1}^{\infty} D_n[\Pi_{j=1}^{n}\beta_j(s_j, y_j)]
\end{aligned}
\tag{12}
$$

where it is noteworthy that the following *re*definition of K does not depend on x:

$$
\begin{aligned}
K(b, s, y; Q) = -\tau\lambda_1(b, y) - [1 - \beta_1(s, y)](s + \tau cy) + \tau\beta_1(s, y)\big\{E\{g(y, D_1) \\
- p[s + \tau g(y, D) - \tau Q]\} - Q\big\}
\end{aligned}
$$

Using the second property, let $(s, y) = (s^*, y^*)$ maximize $K((-s)^+, s, y; Q)$ on its domain. Then the myopic policy $(b_n, s_n, y_n) = ((-s^*)^+, s^*, y^*)$ for all $n = 1, 2, ...$ maximizes $E(B|x_1 = x, w_1 = w)$ for each (x, w) such that $x \leq y^*$. That is, if the initial inventory is no higher than y^*, then the myopic policy is optimal.

In the remainder of this section, we identify s^* and y^* under the following *linearity assumption*: $L(z) = cz$, $\lambda_n(b, \cdot) \equiv \rho b$, $\beta_n(s, y) \equiv \beta \in [0, 1)$, and

$$
p(a) = \theta(-a)^+ \tag{13}
$$

$$
g(y, d) = r\min\{y, d\} - h(y - d)^+ = ry - (r + h)(y - d)^+ \tag{14}
$$

Here, c is a unit cost of acquisition, $\theta > 0$ is a unit default penalty, and ρ is a scalar interest rate. Notice that $p(\cdot)$ is nonincreasing and convex and increases linearly as negative retained earnings decrease. Also, $g(\cdot, d)$ is concave (for each d) and $g(y, \cdot)$ is nondecreasing (for each y).

The following observations are useful to evaluate the partial derivatives of K (from the right):

$$
dE[g(y, D)]/dy = r - (r + h)F(y) \tag{15}
$$

When $s + ry > \tau Q$,

$$\partial E\{p[s + \tau g(y, D) - \tau Q]\}/\partial s = -\theta \mathcal{F} \tag{16}$$

where

$$\mathcal{F} = F[\frac{hy + Q - s/\tau}{h + r}] \tag{17}$$

If $s + ry < \tau Q$, bankruptcy occurs in the next period with probability one. That is, the firm cannot make its coupon payment even if it sells all of its inventory by the end of the period and uses all of its internal capital, so the internal capital in the period that follows must be negative. Therefore, it is reasonable to assume $s + ry > \tau Q$ when assessing the effects of a long-term capital structure on operating decisions.

The following result characterizes the parameter sets yielding an optimal $s < 0$ (so $b > 0$).

Proposition 1. *The firm borrows short-term only when*

$$\tau\beta\theta F(\frac{h\hat{y} + Q}{h + r}) < 1 - \beta - \tau\rho \tag{18}$$

where $\hat{y}$ satisfies

$$(r + h)F(\hat{y}) + \tau\theta h F(\frac{h\hat{y} + Q}{h + r}) = r - c(1 - \beta)/\beta \tag{19}$$

In order to confirm the proposition, use (15) and (16) to obtain

$$\partial K(b, s, y, Q)/\partial y = -\tau c(1 - \beta) + \tau\beta\{r - (r + h)F(y) - \tau\theta h\mathcal{F}\} \tag{20}$$

Setting (20) to zero and evaluating the left-side partial derivative of K with respective to s at $s = 0$ yields

$$(r + h)F(y) + \tau\theta h\mathcal{F} = r - c(1 - \beta)/\beta \tag{21}$$

$$\begin{aligned} \partial K(0, s, y, Q)/\partial s &= -(1 - \beta) - \tau\beta\partial E\{p[0 + \tau g(y, D) - \tau Q]\}/\partial s + \tau\rho \\ &= -(1 - \beta) + \tau\beta\theta F[(hy + Q)/(r + h)] + \tau\rho \end{aligned} \tag{22}$$

Therefore, the firm borrows only when (18) is satisfied.

Equation (18) implicitly restricts $1 - \beta > \tau\rho$, i.e., the gain from receiving a dollar of dividend now rather than next period must be greater than the interest payment for a one-dollar short-term loan. Otherwise, borrowing would never be optimal. In addition, $F[(h\hat{y}+Q)/(h+r)]$ is next period's probability of bankruptcy when $s=0$, so the left side of (18) represents next period's expected bankruptcy cost; the right is the benefit of distributing a dollar dividend now while borrowing the dollar short-term to maintain solvency. Hence, (18) states that the firm should borrow only if the incremental benefit of receiving a dollar of dividend now rather than next period, net of interest payment, is greater than next period's expected default cost.

The following result characterizes y^* and s^* including comparative statics with respect to Q.

Property 1. *(a) The optimal base-stock level* y^*

(i) is invariant with respect to the long-term debt level;

(ii) is nondecreasing in τ *and* ρ *if* $s^* < 0$*;*

(iii) depends only on inventory related parameters if $s^* \geq 0$*;*

(iv) is at least as high when $s^* < 0$ *as when* $s^* \geq 0$*;*

(b) The optimal capital level s^*

(i) is nondecreasing as the long-term debt level increases;

(ii) is nondecreasing in θ, ρ, *and* τ, *and*

(h) depends on the same inventory-related parameters as y^*.

In order to confirm these properties, first consider the case in which (18) does not hold so $b=0$:

$$\begin{aligned} \partial K(0,s,y,Q)/\partial s &= -(1-\beta)+\tau\beta E\{p'[s+\tau g(y,D)-\tau Q]\} \\ &= -(1-\beta)+\tau\beta\theta\mathcal{F} \end{aligned} \tag{23}$$

Setting (23) to zero,

$$\mathcal{F}=(1-\beta)/(\tau\beta\theta) \quad (s\geq 0) \tag{24}$$

Residual retained earnings s balances dividends and retained earnings next period. If the firm increases s by one dollar, then it must defer a one-dollar dividend until next period. However, the bankruptcy cost in the next period is reduced by $\tau\beta\theta$ dollars with probability

$\mathcal{F}$. Hence, (24) indicates that the firm should set its levels of goods and internal capital so that the bankruptcy probability is $(1-\beta)/(\tau\beta\theta)$. Combining (21) and (24),

$$F(y) = \frac{\beta r - (c+h)(1-\beta)}{\beta(r+h)} \tag{25}$$

Define $F^{-1}(u) = sup\{v : F(v) \leq u\}$ for $0 \leq u < 1$, so (25) corresponds to

$$y^* = F^{-1}\Big[\frac{\beta r - (c+h)(1-\beta)}{\beta(r+h)}\Big] \tag{26}$$

Hence, *the optimal base-stock level does not depend on financial parameters when the firm does not borrow.* From (17) and (24),

$$\begin{aligned} s^* &= \tau\Big\{Q + hy^* - (r+h)F^{-1}\Big[\frac{1-\beta}{\tau\beta\theta}\Big]\Big\} \\ &= \tau\Big\{Q + hF^{-1}\Big[\frac{\beta r - (c+h)(1-\beta)}{\beta(r+h)}\Big] - (r+h)F^{-1}\Big[\frac{1-\beta}{\tau\beta\theta}\Big]\Big\} \end{aligned} \tag{27}$$

Although financial parameters do not affect the optimal physical goods base-stock level, the converse is false. *The optimal residual retained earnings depends not only on the firm's long-term debt level and tax rate but also on inventory parameters.*

We now analyze the case in which the firm employs short-term loans to finance its operations, i.e., (18) holds so $s < 0$, and $b > 0$. A similar analysis yields

$$\mathcal{F} = (1-\beta-\tau\rho)/(\tau\beta\theta) \quad (s < 0) \tag{28}$$

$$y^* = F^{-1}\Big[\frac{\beta r - c(1-\beta) - (1-\beta-\tau\rho)h}{\beta(r+h)}\Big] \tag{29}$$

$$s^* = \tau\Big\{Q + hy^* - (r+h)F^{-1}\Big[\frac{1-\beta-\tau\rho}{\tau\beta\theta}\Big]\Big\} \tag{30}$$

That is,

$$s^* = \tau\Big\{Q + hF^{-1}\Big[\frac{\beta r - c(1-\beta) - (1-\beta-\tau\rho)h}{\beta(r+h)}\Big] - (r+h)F^{-1}\Big[\frac{1-\beta-\tau\rho}{\tau\beta\theta}\Big]\Big\}$$

Comparing (28), (29), and (30) with (24), (26), and (27) respectively, one sees that $1-\beta-\tau\rho$ replaces $1-\beta$ when the firm borrows.

In summary, this section reaches three conclusions. First, the market value criterion leads to the same kind of decision rule to manage physical goods inventory as if the criterion were cost minimization. Second, the firm should employ the same kind of decision rule to manage its cash flows as its physical goods, i.e., keeping its residual retained earnings s as close as possible to a target level, s^*. Third, the firm should borrow only when the benefit of receiving a dollar of dividend net of interest payment is greater than the expected bankruptcy cost in the following period. Optimal base-stock levels s^* and y^* cause the probability of bankruptcy in the next period to equal $(1-\beta)/(\tau\beta\theta)$ (if borrowing is optimal, $(1-\beta-\tau\rho)/(\tau\beta\theta)$).

The linearity assumption yields two influences of capital structure on operational policies. First, the optimal physical goods base-stock level does not depend on the capital structure, but the optimal residual internal capital increases as the long-term debt level increases. Second, if the firm borrows short-term, financial parameters (except for the tax rate and short-term interest rate) do not affect the optimal physical goods base-stock level, whereas the optimal residual internal capital depends on both financial and inventory-related parameters. The managerial insight from these results is that maximization of the firm's value is consistent with (i) ignoring financial parameters (other than the tax rate and short-term interest rate) when making short-term operational decisions, but (ii) taking operational parameters into account when making short-term financial decisions. We note that *non*linearity of the default penalty function would invalidate the conclusions that the optimal physical goods base-stock level does not depend on the capital structure, and financial parameters do not affect the physical goods base-stock level.

Numerical Example

To illustrate the importance of coordinating of financial and operational policies, we use the following numerical example. Suppose that $r = 8$, $c = 5$, $h = 4$ per unit per period (quarterly), $\beta = 0.9524$, $\rho = 0.03$, $\tau = 0.8$, $Q = 2$, and each period's demand is normally distributed with mean 25 and standard deviation 10. The monetary loss to the firm should exceed one dollar for each dollar default, so we consider two cases: $\theta = 1$ or $\theta = 5$. A standard production/inventory model maximizes the discouted profit without financial consideration, i.e, the firm maximizes

$$\Pi = \sum_{n=1}^{\infty} \beta^{n-1}[-cz_n + \beta g(y_n, D_n)] \tag{31}$$

subject to $z_n \geq 0$. From (1) and (5), (31) becomes

$$\Pi = cx_1 - \sum_{n=1}^{\infty} cD_n + \sum_{n=1}^{\infty}[-(1-\beta)cy_n + \beta g(y_n, D_n)] \tag{32}$$

Therefore, the optimal base stock level satisfies

$$F(\bar{y*}) = \frac{\beta r - (1-\beta)c}{\beta(r+h)} \tag{33}$$

So, it can be calculated that $\bar{y*} = 35.85$. If $\theta = 1$, then $s^* = -15.443$ and $y^* = 35.7$; if $\theta = 5$, then $s^* = 11.473$ and $y^* = 35.6$. Since the optimal base stock levels are very close, the dividends loss will be small if the firm uses the traditional base stock level $\bar{y*} = 35.85$ while making correct financial decision s^*. On the other hand, if the firm does not use an optimal cash base stock level, for example, setting $s = \tau Q = 1.6$, then the dividends loss could could go as high as 4.3% when $\theta = 1$ and 37.8% when $\theta = 5$.

4 Optimal Capital Structure

This section identifies the level of debt which maximizes the value of the firm, namely the expected present value of dividends. We assume that base-stock levels $y_n = y$ and $s_n = s$ $(n = 1, 2, ...)$ regulate the physical goods inventory and residual retained earnings, and that short-term borrowing is minimal, i.e., $b_n = (-s_n)^+$ for each n. We make no assumption of optimality regarding y and s. As in §4, we do *not* make the linearity assumption, but we do assume that the bankruptcy penalty $p(\cdot)$ is a convex function on the real line. Since the procedure in this section is to differentiate the value of the firm with respect to the coupon payment in order to solve $\partial E(B)/\partial Q = 0$, first we clarify the dependence of $E(B)$ on Q.

It is convenient to assume that the initial inventory x_1 has the same probability distribution as future periods' inventories, so we assume that $x_1 = y - D_0$ where D_0 has the same distribution as D_1 and is independent of $D_1, D_2, ...$. Using definitions (8) and (9) of σ_n^* and $\mathcal{K}_n$, as a consequence of $y_n = y$, $s_n = s$, and $b_n = (-s)^+$ for all n, it follows that $\mathcal{K}_1[(-s)^+, s, y - D, y, Q]$, $\mathcal{K}_2[(-s)^+, s, y - D_1, y,])$, $\mathcal{K}_3[(-s)^+, s, y - D - 2, y, Q]$, ... are independent and identically distributed random variables with the common expected value $E\{K[(-s)^+, s, y - D_0, y, Q]\}$. Therefore, from (9), B has the same probability distribution as $\sigma_\infty^* \mathcal{K}_1[(-s)^+, s, y - D, y, Q] + w_1 - p(w_1)$. Since $w_1 = mQ + \eta \geq 0$ where η is the amount of equity and mQ is the initial capital associated with the coupon payment of Q each period, we assume that $p(w_1) = 0$. Therefore,

$$E(B) = E\{K[(-s)^+, s, y - D_1, y, Q]\}/(1 - \gamma) + mQ + \eta \tag{34}$$

where

$$\gamma = [E(\sigma_\infty^*) - 1]/E(\sigma_\infty^*) \tag{35}$$

That is, γ is the deterministic single-period discount factor whose present value of a \$1 perpetual annuity equals $E(\sigma_\infty^*)$.

Assuming in (34) that the derivative of the expected value is the expected value of the derivative,

$$(1 - \gamma)\partial E(B)/\partial Q = m(1 - \gamma) + \partial E\{K[(-s)^+, s, y - D_1, y, Q]\}/\partial Q \tag{36}$$

Let $\omega(s, y) = E[\beta_1(s, y)]$. From (9), (10), and the independence of $\beta_n(s, y)$ and $g(y, D_n)$,

$$\partial E\{K[(-s)^+, s, y - D_1, y, Q]\}/\partial Q = \tau\omega(s, y)E\big(\tau p'[s + \tau g(y, D_1) - \tau Q] - 1\big) \tag{37}$$

Therefore, the convexity of $p(\cdot)$ ensures

$$(1 - \gamma)\partial^2 E(B)/\partial Q^2 = -\tau^2\omega(s, y)E\big(p''[s + \tau g(y, D_1) - \tau Q]\big) \leq 0 \tag{38}$$

So long-term debt in the amount mQ maximizes the value of the firm if $\partial E(B)/\partial Q = 0$, and this first-order condition corresponds to

$$E\big(p'[s + \tau g(y, D_1) - \tau Q]\big) = 1/\tau - m[1 - \gamma(s, y)]/[\tau^2\omega(s, y)] \tag{39}$$

The following property is apparent in (39).

Property 2. *If the bankruptcy penalty is a convex function, then the capital structure that maximizes the value of the firm does not depend on short-term interest or inventory replenishment cost ($\lambda_n(\cdot,\cdot)$) or $L(\cdot)$).*

There are clear consequences of (39) when the linearity assumption is in force, and demand has a positive density function (where the distribution function of demand is $0 < F(\cdot) < 1$). Then $0 < \alpha < 1$ implies that there is a unique number $F^{-1}(\alpha)$ such that $F[F^{-1}(\alpha)] = \alpha$, $p'(a) = -\theta$ if $a < 0$, and $p'(a) = 0$ if $a > 0$. Hence, the left side of (39) is

$$\begin{aligned} E\big(p'[s + \tau g(y, D_1) - \tau Q]\big) &= -\theta P\{s + \tau r y - \tau(r + h)(y - D_1)^+ \leq \tau Q\} \\ &= -\theta F\big[(hy - s + \tau Q)/[\tau(r + h)]\big] \end{aligned}$$

Substituting this expression in (39),

$$F\big[(hy - s + \tau Q)/[\tau(r + h)]\big] = m[1 - \gamma(s, y)]/[\theta\tau^2\omega(s, y)] - 1/(\tau\theta) \tag{40}$$

Take F^{-1} on both sides and rearrange terms to obtain the following special case of (39):

$$Q = (s - hy)/\tau + (r + h)F^{-1}\big\{\big(m[1 - \gamma(s, y)]/[\omega(s, y)\tau] - 1\big)/(\tau\theta)\big\} \tag{41}$$

We note that (41) resembles (26) and (29) with $\beta_n(s, y) \equiv \beta \in [0, 1)$.

All-debt financing is optimal if $\partial E(B)/\partial Q > 0$ for all Q, i.e., if

$$\theta F\big[(hy - s + \tau Q)/[\tau(r + h)]\big] < m[1 - \gamma(s, y)]/[\tau^2\omega(s, y)] - 1/\tau$$

for all Q. This condition corresponds to $1 < m[1 - \gamma(s, y)]/[\theta\omega(s, y)\tau^2] - 1/(\tau\theta)$ or, defining $A(s, y) = m[1 - \gamma(s, y)]/[\omega(s, y)\tau]$,

$$A(s, y) > 1 + \theta\tau \tag{42}$$

Similarly, a sufficient condition for all-equity financing is

$$A(s, y) < 1 \tag{43}$$

The interval in which a mixture of long-term debt and equity is optimal, $1 < A(s, y) < 1 + \theta\tau$, grows with the unit cost of default (θ) and diminishes with the tax rate ($1 - \tau$). It is striking that neither (42) nor (43) depends on a production-inventory parameter.

Property 3. *Under the linearity assumption, if demand has a positive density function, then only financial parameters are sufficient to determine whether an optimal capital structure should be all-debt or all-equity.*

5 Variances of Dividends and Profits

Among public U.S. corporations, the variance of dividends is less than the variance of profits (Allen and Winton (1995)). This *smoothing* effect is a property of the dividends and profits in the model in §2 if the firm employs a base-stock policy for selecting levels of physical goods supply and residual retained earnings. Although such a policy is optimal if $L(\cdot)$ is linear (Li et al. (1997)), we do not make the linearity assumption in this section. Moreover, the variances do not depend on the firm's long-term debt level.

We assume that physical inventories and residual retained earnings are regulated with base-stock levels y and s, respectively, and initial inventory is low enough to permit $z_1 = y - x_1$. That is, for all $n \geq 1$, $y_n = y$ and $s_n = s$ for target levels y and s. The conclusions do not depend on whether y and s are optimal in any sense.

There is ambiguity in the definitions of "profit" and "dividend" because the former could be pre-tax or post-tax, and the latter could include or suppress capital subscriptions. Let J_n be the pre-tax profit in period n, so the post-tax profit is τJ_n, and for brevity let $g_n = g(y, D_n)$, $L_n = L(y - x_n)$, and $\lambda_n = \lambda_n([-(s)^+, y]$. Using (6), the pre-tax profit is

$$J_n = g_n - L_n - \lambda_n - p(w_n) = g_n - L_n - \lambda_n - p(s + \tau g_{n-1} - \tau Q) \tag{44}$$

From (2), (6), and (44), the dividend net of capital subscription is

$$v_n = w_n - s - \tau[L_{n-1} + \lambda_n + p(w_n)] = \tau(g_{n-1} - g_n - Q + J_n) \tag{45}$$

and $[(v_n)^+]$ is the dividend with capital subscription suppressed. Let $Var(X)$ denote the variance of the random variable X and let $Cov(X, Y)$ denote the covariance of X and Y. This leads to four possible comparisons of variances: $Var(J_n) - Var(v_n)$, $Var(J_n) - Var[(v_n)^+]$, $Var(\tau J_n) - Var(v_n)$, and $Var(\tau J_n) - Var[(v_n)^+]$. The smallest of these differences is

$Var(\tau J_n) - Var(v_n)$ because $0 \leq \tau \leq 1$ and $Var[(v_n)^+] \leq Var(v_n)$. This discussion glosses over the details of carry-forward tax credits for operating losses.

Proposition 2. *Suppose that $p(\cdot)$ is nonincreasing, $L(\cdot)$ is nondecreasing, and $g(y, \cdot)$ is nondecreasing (for any y). If a firm regulates physical goods inventory and residual retained earnings with base-stock levels, then the variance of the dividend is no greater than the variance of the profit.*

The structures in (13) and (14) exemplify the assumptions because $p(\cdot)$ is nonincreasing and $g(y, \cdot)$ is nondecreasing (for any y) .

To confirm the proposition, from (45),

$$\tau J_n = v_n + \tau(g_n - g_{n-1} + Q)$$

Since $Cov(J_n, g_n) = 0$,

$$Var(\tau J_n) = Var(v_n) + 2\tau^2 Var(g_1) - 2\tau Cov(J_n, g_{n-1}) \tag{46}$$

where

$$Cov(J_n, g_{n-1}) = Cov[p(s + \tau g_{n-1}, g_{n-1})] = Cov\{p[s + \tau g(y, D_{n-1})], g(y, D_{n-1})\}$$

which is non-positive due to the monotonicity assumptions and (Esary, Proschan and Walkup (1967)).

It follows from (46) that the firm's leverage does not affect the variances.

6 Extensions

The general model can be extended in various ways while preserving key conclusions. For example, results similar to Proposition 1 and Property 2 can be obtained for a model with multiple products having inter-dependent inventory replenishment costs but independent demands.

If the model is changed so the firm is liquidated after its first default, i.e., if *wipeout* bankruptcy replaces reorganization bankruptcy, then Properties 1 and 3 are no longer valid.

It seems that wipeout bankruptcy corresponds to a default penalty that is so severe that the firm cannot offet the risk induced by a higher debt level only by adusting its cash level. Therefore, the capital structure, short-term financial policy, and inventory policy are more interdependent than with reorganization bankruptcy.

7 Summary

Most studies of capital structure are devoid of operational detail, and capital structure is absent from most studies of operational decisions. The financial studies emphasize agency issues or the tax advantage of long-term debt, while models of production and inventory phenomena lack financial constraints. We analyze a model of short-term operational and financial decisions in which the long-term capital structure is explicit. This leads to conditions that permit a value-maximizing firm to detach its short-term operational decisions from financial considerations, but its short-term cash flow management should reflect its production and inventory decisions.

Given the type of policy for short-term decisions that turns out to be optimal, we ask two questions. First, what is the value-maximizing long-term capital structure? The optimal mixture of equity and long-term debt depends on some (but not all) of the parameters of the day-to-day production-inventory system. However, sufficient conditions for the optimal mixture to be all-equity or all-long-term debt depend only on financial parameters.

Second, how does the variability of dividends compare with the variability of profits? We show that the variance of dividends is less than the variance of profits. This is consistent with the empirical phenomenon known as the *smoothing effect*.

We do not examine natural extensions of this work such as multiple products, dynamic adjustment of the product selling price, and competitive considerations.

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