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The Family Periodic Loading Problem

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Abstract

The Multiple Family Economic Lot scheduling Problem with safety stocks (MFELSP-SS) with normally distributed, time-stationary demand is considered in a manufacturing setting where the relevant costs include family setup costs, item setup costs, and inventory holding costs for both cycle and safety stocks. The Family Planning Problem (FPP) is the first step in addressing the MFELSP—SS. The solution to the FPP is comprised of the basic period length, the family multipliers, and the item multipliers that give the lowest total cost of setups and carrying inventory. The family multipliers and items multipliers are restricted to integer powers of two. The solution to the FPP is used as the input to the Family Periodic Loading Problem (FPLP), which is the subject of this paper. The purpose of the FPLP is to create a feasible production schedule that achieves, as much as possible, the cost minimizing objective of the FPP.

The FPLP generates a production schedule that can be implemented, a feature that is not present in the FPP—or in many ELSP solution procedures. Three efficient heuristic approaches to solving the FPLP are implemented and a comparison of their performance is presented.

Subject classification: 334 multiproduct lot sizing, MFELSP—SS: feasibility issues

1. Introduction

In this paper, we take the solution of the Family Planning Problem (FPP) (Karalli and Flowers, 2004) as the starting point of scheduling the families and items on a single machine. The Multiple Family Economic Lot Scheduling Problem with Safety Stocks (MFELSP—SS) is an important problem that regularly occurs in practice that has not been adequately addressed in the literature. In particular, it is common that production is to stock in such situations, rather than to customer order. Because of substantial setup times, it is not possible to setup the machine for production to customer order. Thus, safety stocks are required to buffer against demand uncertainty between planned runs of a particular product within a particular family. Because of the need for safety stocks in these situations, we have adopted a basic period approach to its solution (Karalli and Flowers, 2004) rather than a time-varying lot sizing approach. In addition, Just-in-Time manufacturing systems favor a basic period approach for coordinating with suppliers.

The MFELSP-SS may be more completely characterized as a continuous-time, infinite-horizon extension of the ELSP where

N families, each having N items, are produced in the same facility, one unit at a time.

For each family, there is a:

- sequence-independent setup cost
- sequence-independent setup time

For each item, the demand is:

- time-stationary
- normally distributed, with
- known mean and standard deviation
- uncorrelated with other items
- not substitutable with other items.

For each item:

- there is a known, constant production rate
- there is a sequence-independent setup cost
- there is a sequence-independent setup time
- there is a specified customer service level

- safety stock is maintained in order to meet the specified service level, and
- there is no backlogging.

For notational convenience, we take N to be both the number of families and items within each family, without loss of generality (w.l.o.g.). When the number N exceeds the actual number of items in any family, or exceeds the number of families, we simply create dummy items and/or families and assign the value of zero to their parameters.

The solution to the FPP of N families with N items in each family is a basic period length, T, an N-vector of family multipliers, $\boldsymbol{K}$, and an $\boldsymbol{N}\times\boldsymbol{N}$ – matrix of item multipliers, $\boldsymbol{k}$. The solution, $(T, \boldsymbol{K}, \boldsymbol{k})$, specifies low cost production intervals for each item produced on the single machine. The production intervals, in turn, determine the required working and safety stock levels.

The solution to the FPP, however, does not specify the production sequence of the families and items, nor does it guarantee a feasible schedule given the scheduling assumptions under which the solution is generated. The feasibility issue is not unique to the MFELSP—SS; in fact, it is systemic to any approach to the ELSP where multipliers are generated in a model and setup times are only considered in constraints as necessary (but not sufficient) conditions for feasibility.

Whereas the mathematical formulation of the FPP, as well as ELSP formulations such as those referenced above, imposes a structure of equal lot sizes, termed artificial by Dobson (1987), it lends itself to cyclic schedules as does the FPLP. As such, the Family Periodic Loading Problem (FPLP) begins with $(T, \boldsymbol{K}, \boldsymbol{k})$ as its problem input and generates a solution in the form of a vector, $\boldsymbol{f}$, which specifies the order of the items to be produced. For each item, near-optimal production and idle times will be computed.

Maxwell (1964) considered a similar approach in his seminal paper. Dobson (1987) provides a procedure for the deterministic ELSP, whereby $\boldsymbol{f}$ is specified and cost minimizing production times and idle times are generated. Zipkin (1991) takes $\boldsymbol{f}$ as given and provides a solution approach to generating cost-minimizing production and idle times for the deterministic ELSP. Our approach considers production run times, and both family and item setup times in generating a cyclic schedule where the minimum idle time across all loads is maximized.

Schweitzer et al. (1988) shows that the periodic loading problem is NP-Complete. It is therefore necessary to develop an efficient procedure to create a feasible cyclic schedule when one exists. Gallego (1990) provides a dynamic control procedure whereby there always exists an f-recovery schedule. The implication for the FPLP is that it can be implemented in a random-demand environment.

Dobson (1987) allows for unequal lot sizes to be scheduled for any item. This relaxation of the equal lot size rule allows for the Periodic Loading Problem to be feasible when the sum of the ratios of each item's demand rate to production rate is less than one. In this paper, we enforce the equal lot size rule because of the existence of safety stocks. Allowing the lot sizes to vary would further complicate the problem by requiring the safety stock levels to vary as well. We assume that, in most cases, every setup and production run can be scheduled according to $(T, \boldsymbol{K}, \boldsymbol{k})$. This assumption is reasonable when we consider that in practice, firms tend to operate with excess capacity in order to both meet future growth in demand for current products and to introduce new products. Such excess capacity may also be used to accommodate schedules exhibiting uneven capacity utilization.

In Section 2, we define and discuss the properties of the FPLP. In Section 3, we present a formulation for the FPLP. In Section 4, efficient heuristics for loading the basic periods are proposed. In Section 5, we present the Lagrangian relaxation used to test our heuristics. In Section 6, we provide computational results, followed by concluding remarks in Section 7.

2. The FPLP

The purpose of the FPLP is to generate a feasible schedule such as the one in Figure 1 using $(T, \boldsymbol{K}, \boldsymbol{k})$ as the input. The variable T represents the length of the basic period as determined by the FPP (Karalli and Flowers 2004). The production interval for every family $i \in \boldsymbol{N}$ is in K_i multiples of the basic period. The production interval for every item $(i, j) \in \boldsymbol{N} \times \boldsymbol{N}$ is in k_{ij} multiples of the basic period. In the sample schedule in Figure 1, family 1 has a multiplier value of $K_1 = 1$, which means it is produced every basic period. Families 2 and 3 both have family multipliers of 2, so they are produced every other period (Family 2 in odd numbered periods, Family 3 in even). Item $(1, 1)$ has a multiplier of $k_{11} = 1$, so it is produced every time that family 1 is produced. Items $(1, 2)$ and $(1, 3)$ both have multiplier values of 2, so they are produced only every other time that family 1 is produced. Items $(2, 1)$ and $(2, 3)$ both have a multiplier value of 2, indicating each is produced every other time family 2 is setup. Item $(2, 2)$ has a multiplier value of $k_{22} = 1$. It is produced every time family 2 is setup. Similar interpretations may be gleaned for the products of family 3 from Figure 1. P_{ij} is the production time for item (i, j).

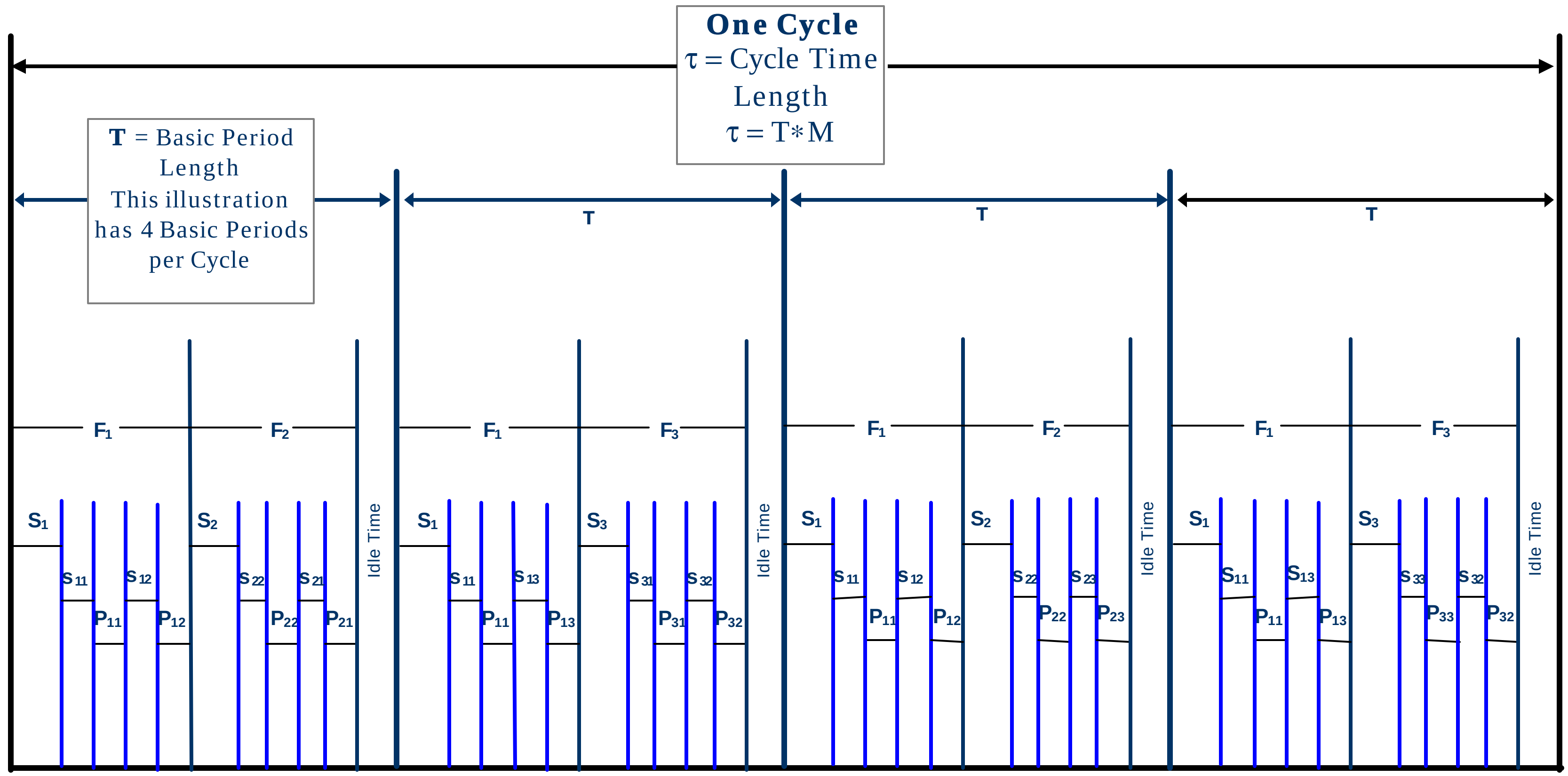

Figure 1: Cycle time, Basic Period, and Multipliers for a sample MFELSP-SS problem.

3. A Formulation of the FPLP

Given the solution to the FPP, $(T, \boldsymbol{K}, \boldsymbol{k})$, we seek to create a feasible cyclic schedule. The problem environment is characterized as follows:

N Families, each having N items, need to be scheduled for production in a single facility.

The cyclic schedule comprises M basic periods of length T, where:

- T is provided in the solution to the FPP (Karalli and Flowers, 2004)
- The size of M depends on $(T, \boldsymbol{K}, \boldsymbol{k})$, M is an input into the problem

The cyclic schedule is τ time units long.

For each family there is:

- a known family multiplier—solved for in the FPP
- a sequence-independent setup time

For each item there is:

- a known item multiplier—solved for in the FPP
- a sequence-independent setup time,
- a production lot size, dependent on $(T, \boldsymbol{K}, \boldsymbol{k})$

The following parameters are inputs to the problem:

S_i	Setup time for family i
s_{ij}	Setup time for the j^{th} item in family i
d_{ij}	Demand mean for item j in family i
s_{ij}^2	Demand variance for item j in family i
p_{ij}	Production rate for item j in family i
r_{ij}	$\rho_{ij} = d_{ij}/p_{ij}$
r	$\rho = \sum_{i \in \mathbf{N}} \sum_{j \in \mathbf{N}} (d_{ij}/p_{ij})$

We introduce the superscript t, which indexes the basic period sequence number in the cycle, $t = 1, \cdots, M$.

The solution to the FPP is used to compute the following:

M Number of basic periods in a cycle. $M = \boldsymbol{max}\left\{K_i \cdot k_{ij} : i, j \in \boldsymbol{N}\right\}$

$\boldsymbol{t}$ $\boldsymbol{t} = T \cdot M$

P_{ij} Production run time, including item setup time, for family *i*, item *j*. $P_{ij} = s_{ij} + \boldsymbol{r}_{ij} \cdot T \cdot K_i \cdot k_{ij}$

F_i^t Total time for producing family i in period t. $F_i^t = S_i \cdot x_i^t + \sum_{j=1}^{N} P_{ij} \cdot y_{ij}^t$. The decision variables x_i^t and y_{ij}^t are defined next.

The decision variables of the FPLP are:

x_i^t 0-1 decision variable. $x_i^t = 1$ if family *i* is produced in period *t*; i $\boldsymbol{N}$; $x_i^t = 0$ otherwise

y_{ij}^t 0-1 decision variable. $y_{ij}^t = 1$ if item (i, j) is produced in period *t*; $(i, j) \in \boldsymbol{N} \times \boldsymbol{N}$; $y_{ij}^t = 0$ otherwise

Define:

$\boldsymbol{X}$ $\boldsymbol{X} = \left\{x_i^t : i \in \boldsymbol{N}, t = 1, \cdots, M\right\}$

$\boldsymbol{Y}$ $\boldsymbol{Y} = \left\{y_{ij}^t : (i, j) \in \boldsymbol{N} \times \boldsymbol{N}, t = 1, \cdots, M\right\}$

b $b = (t-1)\ \boldsymbol{mod}\ K_i + 1$

q $q = (t-1)\ \boldsymbol{mod}\ k_{ij} + 1$

k_{ij} $k_{ij} = K_i \cdot k_{ij}$

$\boldsymbol{I}$ $\boldsymbol{I} = \{i : i \in \boldsymbol{N}, K_i \geq 2\}$, the set of family indices for which the family multiplier value is $K_i \geq 2$

$\boldsymbol{J}$ $\boldsymbol{J} = \left\{(i, j) : (i, j) \in \boldsymbol{N} \times \boldsymbol{N}, k_{ij} \geq 2\right\}$, the set of items $(i, j) \in \boldsymbol{N} \times \boldsymbol{N}$ for whom $k_{ij} \geq 2$

$\#\boldsymbol{I}$ The cardinality of $\boldsymbol{I}$

$\#\boldsymbol{J}$ The cardinality of $\boldsymbol{J}$

$\boldsymbol{R}$ The set of real numbers

T_t Capacity remaining in basic period t, $t = 1, \ldots, M$

The objective of Problem S, the formulation given to solve the FPLP, is to maximize the minimal idle time, L, resulting from the periodic loading process; that is, find L, $\boldsymbol{X}$, and $\boldsymbol{Y}$ so as to

$$\textit{Maximize} \quad L$$

$$\text{Subject to} \quad \sum_{i=1}^{N} x_i^b \cdot S_i + \sum_{i-1}^{N} \sum_{j=1}^{N} y_{ij}^q \cdot P_{ij} + L \le T_t; \quad \forall\ t = 1, \cdots, M \tag{1}$$

$$\sum_{t=1}^{M/K_i} x_i^t = 1; \qquad \forall\ i = 1, \cdots, N \tag{2}$$

$$\sum_{t=1}^{M/k_{ij}} y_{ij}^t = 1; \qquad \forall\ i, j = 1, \cdots, N \tag{3}$$

$$\sum_{j=1}^{N} y_{ij}^t - N \cdot x_i^t = 0; \qquad \forall\ i = 1, \cdots, N; \forall\ t = 1, \cdots, K_i \tag{4}$$

$$x_i^t, y_{ij}^t \sim 0-1; \qquad \forall\ i, j = 1, \cdots, N; \forall\ t = 1, \cdots, M \tag{5}$$

$$L \ge 0 \tag{6}$$

Condition (1) is a capacity constraint on the basic period length. Condition (2) allows for the selection of only one of the M/K_i possible family i schedules that are in accordance with the frequencies indicated by the family multipliers. Condition (3) allows for the selection of only one of the M/k_{ij} possible item (i, j) schedules that are in accordance with the frequencies indicated by the item multipliers. Condition (4) enforces the technical requirement that items be scheduled only when their families have been setup.

The formulation above results in an integer program with n variables and p constraints, where

$$n = 1 + \sum_{\substack{i-1 \\ K_i \geq 2}}^{N} K_i + \sum_{\substack{(i,j) \in \boldsymbol{N} \times \boldsymbol{N} \\ k_{ij} \geq 2}} k_{ij} \tag{7}$$

$$p = M + \# \boldsymbol{I} + \# \boldsymbol{J} + \sum_{\substack{i=1 \\ K_i \geq 2}}^{N} K_i . \tag{8}$$

We account for the minimum slack time variable by the number 1 on the RHS of (7). The second term of the RHS of (7) accounts for each of the different starting positions in the M basic periods that each family can be scheduled. We need one variable for each potential starting position. Similarly, the third term on the RHS of (7) accounts for each of the different starting positions in the M basic periods that each item can be scheduled, that is every k_{ij}.

There are M capacity constraints, one for each basic period. The set $\boldsymbol{I} = \{ i : i \in \boldsymbol{N}, K_i \geq 2 \}$ is the set of family indices for which the family multiplier value is $K_i \geq 2$. The set $\boldsymbol{J} = \{ (i, j) : (i, j) \in \boldsymbol{N} \times \boldsymbol{N}, k_{ij} \geq 2 \}$ is the set of items $(i, j) \in \boldsymbol{N} \times \boldsymbol{N}$ for which $k_{ij} \geq 2$.

The reader will note that the families with multiplier values $K_i = 1$ and the items for which $k_{ij} = 1$ are omitted from the mixed integer program (MIP). The reason for this is that these families and items are scheduled in every basic period, affording no flexibility. The capacities of each basic period need to be adjusted to reflect the fact that the omitted families and items have been scheduled. For each family whose multiplier values $K_i = 1,$ adjust the capacity by subtracting S_i from the available capacity. For each item (i, j) for which $k_{ij} = 1,$ subtract P_{ij} from the available capacity, where

$$P_{ij} = s_{ij} + r_{ij} \cdot T \cdot K_i \cdot k_{ij}. \qquad (9)$$

The item setup time is s_{ij}, and the item production time is $r_{ij} \cdot T \cdot K_i \cdot k_{ij}$.

Karalli and Flowers (2004) show that any solution to the MFELSP—SS can be represented in anchor form (AF). That is, for every solution $(T, \boldsymbol{K}, \boldsymbol{k})$ to the FPP, the following conditions hold:

- $0 < T \in \boldsymbol{R}$
- $\boldsymbol{K} = (1, K_2, \cdots, K_N)$; with $1 \le K_2 \le \cdots \le K_N$
- $\forall\, i \in \boldsymbol{N}, K_i \in \boldsymbol{P},$ where $\boldsymbol{P} = \left\{2^P : p \in \boldsymbol{Z}_+\right\}$
- $\boldsymbol{k}$ is an $(\boldsymbol{N} \times \boldsymbol{N})$ – matrix whose i^{th} row, $k_{i\cdot}$, is the vector of item multipliers for family i
- $k_{i\bullet} = (1, k_{i2}, \cdots, k_{iN})$; with $1 \le k_{i2} \le \cdots \le k_{iN}$
- $\forall\ (i, j) \in \boldsymbol{N} \times \boldsymbol{N}, k_{ij} \in \boldsymbol{P}$

Because the smallest family and item multiplier values are always equal to 1, we are always able to omit the associated family and item variables for which $k_{ij} = 1,$ and the associated family selection (Equation (2)) and technical constraints (Equation (4)) as well.

4. Heuristics

We consider three heuristics for solving this problem. The Preparation Stage below is a heuristic determination of family i^*, which has the potential of having the most uneven capacity utilization across the M periods. The implication of uneven capacity utilization is that there may be one or more periods where a large portion of its capacity is required for family i^* and its items. Scheduling these first would potentially avert any problems that would arise if they were scheduled later with not enough capacity in any single period. The following steps are a preparation for the three heuristics.

Preparation Stage:

Generate Problem S as in Section 3.

Step 1: Adjust the basic period capacities:

$\forall\, i \in N,$ if $K_i = 1,$ then $T_t \leftarrow T_t - S_i, \forall t = 1, \ldots, M$

$\forall\, (i, j) \in N \times N,$ if $k_{ij} = 1,$ then $T_t \leftarrow T_t - P_{ij}, \forall t = 1, \ldots, M$

Step 2: Compute $J_i, \forall j \in \boldsymbol{N}$.

$$\forall\, i \in \boldsymbol{N}, \text{ let } J_i = \begin{cases} 0 & \text{if } K_i = 1 \\ \sum_{j=1}^{N} \frac{k_{ij}}{K_i} & \text{if } K_i \geq 2 \end{cases}$$

Step 3: Select i* so that $(K_{i*}, J_{i*}) \geq^{L} (K_i, J_i), \forall\, \text{i} \in \boldsymbol{N}$.

We can now rewrite Problem S as follows, and we will call this version Problem S':

Maximize L

Subject to
$$\sum_{i=1}^{N} x_i^b \cdot S_i + \sum_{i-1}^{N} \sum_{j=1}^{N} y_{ij}^q \cdot P_{ij} + L \leq T_t; \quad \forall\, t = 1, \cdots, M \qquad (10)$$

$$\sum_{t=1}^{M / K_i} x_i^t = 1; \qquad \forall\, i^* \neq i = 1, \cdots, N \qquad (11)$$

$$\sum_{t=1}^{M/K_i} x_{i*}^t = 1 \tag{12}$$

$$\sum_{t=1}^{M/k_{ij}} y_{ij}^t = 1; \qquad \forall\ i,\ j = 1,\cdots, N;\ i* \neq i \tag{13}$$

$$\sum_{t=1}^{M/k_{ij}} y_{i*j}^t = 1 \tag{14}$$

$$\sum_{j=1}^{N} y_{ij}^t - N \cdot x_i^t = 0; \qquad \forall\ i \in \boldsymbol{N} \setminus \{i*\}; \forall\ t = 1,\cdots,(K_i - 1) \tag{15}$$

$$\sum_{j=1}^{N} y_{i*j}^t - N \cdot x_{i*}^t = 0; \qquad \forall\ t = 1,\cdots,(K_i - 1) \tag{16}$$

$$x_i^t, y_{ij}^t \sim 0-1; \qquad \forall\ i, j = 1,\cdots, N; \forall\ t = 1,\cdots, M \tag{17}$$

$$L \geq 0 \tag{18}$$

The following heuristics will involve the removal, and return of columns and rows of Problem S'. We will always refer to the constraints of the MIP (even when modifications are made) as $\boldsymbol{A}x = \boldsymbol{b}$. We also refer to the column vector under a variable $x \in \boldsymbol{x}$ by $\boldsymbol{a}(x)$.

Each constraint in (11) and (12) can uniquely be identified by the family under consideration, and in (13) and (14) by the item. More precisely, we refer to constraint (12) as constraint x_{i*} and more generally to each constraint in (11) as constraint x_i. Similarly, we refer to constraint (14) as constraint y_{i*j}, and each constraint in (13) as constraint y_{ij}. Each family technical constraint in (15) and

(16) is identifiable by the family under consideration and by the family's starting period. We refer to constraint (16) as constraint Φ_{i*}^{t} and more generally to each constraint in (15) as constraint Φ_{i}^{t}.

H1:

Heuristic H1 was initially motivated by an approach to solving the traveling salesperson problem, whereby the relaxed assignment problem is first solved and tour breaking constraints are subsequently added until a feasible tour is obtained. The approach is modified as follows: In H1, remove all the technical constraints in (15) above. That is, constraints $\Phi_{i}^{t}, t = 1, \cdots, (K_i - 1)$, $i* \neq i = 1, \cdots, N$, are removed from the MIP. Family $i*$ is selected for having the potential for the most uneven capacity utilization across the M basic periods. Next run the first-stage MIP. Then, adjust the capacities of each period to reflect the scheduling of family $i*$ and its items. Taking the scheduling of family $i*$ as fixed, remove the columns for family $i*$ and its items from the MIP. Remove constraints x_{i*}, y_{i*j}, and Φ_{i*}, since they are no longer needed.

In stage 2, return the constraints, removed in stage 1, to the MIP—resulting in a smaller problem than the original. The solution to the second-stage MIP, augmented by the schedule previously obtained for family $i*$, is the schedule for the MFELSP—SS.

Stage 1: Remove constraint $\Phi_{i}^{t}, t = 1, \cdots, (K_i - 1)$, $i* \neq i = 1, \cdots, N$, from Problem S. Solve the modified problem.

$\forall\ t = 1, \cdots, (K_{i*} - 1)$, if $x_{i*}^{t} = 1$ then $b \leftarrow b - \boldsymbol{a}\left(x_{i*}^{t}\right)$.

$\forall\ t = 1, \cdots, (K_i - 1),$ and $\forall\ j = 1, \cdots, N,$ if $y^t_{i*j} = 1$ then $b \leftarrow b - \boldsymbol{a}\left(y^t_{i*j}\right).$

Remove the following columns:

$\boldsymbol{a}\left(x^t_{i*}\right), \forall\ t = 1, \cdots, K_{i*},$ and

$\boldsymbol{a}\left(y^t_{i*j}\right), \forall\ t = 1, \cdots, K_{i*}, \forall\ j = 1, \cdots, N$

Remove the following rows:

x_{i*}

$y_{i*j}, \forall\ j = 1, \cdots, N,$ and

$\Phi^t_{i*}, \forall\ t = 1, \cdots, (K_{i*} - 1)$

Stage 2: Add back constraint(s) $\Phi^t_i, \forall\ t = 1, \cdots, (K_i - 1),\ i* \neq i = 1, \cdots, N,$ to the MIP.

Solve the adjusted problem.

The solution to the adjusted problem is the production schedule for the MFELSP—SS.

<u>H2:</u>

For heuristic H2, proceed with Stage 1 by removing all of the y^t_{ij} columns, $\forall\ i \neq i*,$ from the MIP. Remove all associated constraints; that is, the y_{ij}'s and Φ^t_i's. Next, we solve the modified MIP. The solution to the Stage 1 MIP includes the schedule for family $i*,$ which will be part of the final solution of the FPLP. Stage 1 is completed by adjusting each of the M period capacities in order to reflect the scheduling positions of family $i*$ and its items, followed by removing the x_{i*}^{t} and y_{i*j}^{t} columns and constraints x_{i*}, y_{i*j}, and Φ_{i*}, since they are no longer needed.

In the second stage of H2, return all of the y_{ij}^{t} columns, $\forall\, i \neq i^{*}$, to the MIP. Return all associated constraints; that is the y_{ij}'s and Φ_{i}^{t}'s. Then solve the resulting MIP. The solution to the second-stage MIP, augmented by the schedule previously obtained for family $i*$ in stage 1, is the schedule for the MFELSP—SS.

Stage 1: Solve:

Maximize L

Subject to
$$x_{i*}^{b} \cdot S_{i*} + \sum_{j=1}^{N} y_{i*j}^{q} \cdot P_{i*j} + L \leq T_t; \quad \forall\, t = 1, \cdots, M \tag{19}$$

$$\sum_{t=1}^{M/K_i} x_{i*}^{t} = 1 \tag{20}$$

$$\sum_{t=1}^{M/K_i} x_{i}^{t} = 1; \qquad \forall\, i* \neq i = 1, \cdots, N \tag{21}$$

$$\sum_{t=1}^{M/k_{ij}} y_{i*j}^{t} = 1 \tag{22}$$

$$\sum_{j=1}^{N} y_{i*j}^{t} - N \cdot x_{i*}^{t} = 0; \qquad \forall\, t = 1, \cdots, (K_i - 1) \tag{23}$$

$$x_{i}^{t}, y_{ij}^{t} \sim 0-1; \qquad \forall\, i, j = 1, \cdots, N; \forall\, t = 1, \cdots, M \tag{24}$$

$$L \geq 0 \tag{25}$$

Next,

$$\forall\ t = 1, \cdots, (K_{i*} - 1),\ \text{if } x_{i*}^t = 1 \text{ then } b \leftarrow b - \boldsymbol{a}\left(x_{i*}^t\right).$$

$$\forall\ t = 1, \cdots, (K_i - 1),\ \text{and } \forall\ j = 1, \cdots, N,\ \text{if } y_{i*j}^t = 1 \text{ then } b \leftarrow b - \boldsymbol{a}\left(y_{i*j}^t\right).$$

The last step (just above) reduces the remaining available capacities of the M basic periods, to account for the scheduling positions of family $i*$ and its items when they are removed from the MIP for the second stage. To consider the fact that the RHS values of the M constraints in (19) are no longer equal to T, we refer to these values as $T_1, T_2, \cdots, T_M$.

Stage 2: Solve:

Maximize L

Subject to
$$\sum_{\substack{i=1 \\ i \neq i*}}^{N} x_i^b \cdot S_i + \sum_{\substack{i-1 \\ i \neq i*}}^{N} \sum_{j=1}^{N} y_{ij}^q \cdot P_{ij} + L \leq T_t; \qquad \forall\ t = 1, \cdots, M \tag{26}$$

$$\sum_{t=1}^{M / K_i} x_i^t = 1; \qquad \forall\ i* \neq i = 1, \cdots, N \tag{27}$$

$$\sum_{t=1}^{M / k_{ij}} y_{ij}^t = 1; \qquad \forall\ i, j = 1, \cdots, N; i* \neq i \tag{28}$$

$$\sum_{j=1}^{N} y_{ij}^t - N \cdot x_i^t = 0; \qquad \forall\ i \in \boldsymbol{N} \setminus \{i*\}; \forall\ t = 1, \cdots, (K_i - 1) \tag{29}$$

$$x_i^t, y_{ij}^t \sim 0-1; \qquad \forall\ i, j = 1, \cdots, N; \forall\ t = 1, \cdots, M \tag{30}$$

$$L \geq 0 \tag{31}$$

The solution to the adjusted problem is the production schedule for the MFELSP—SS.

H3:

The strategy for heuristic H3 is to solve the FPLP one family at a time. This approach may be desirable for large-scale problems. The first stage is similar to that of H2, but Stage 2 proceeds by repeating Stage 1 of H2 for $N-1$ families. Repeat Stage 2 until one family remains. H3 terminates when each family's schedule is obtained.

Stage 1: Proceed with the first stage of heuristic H3 as is done for H2. The subsequent stages of H3 differ.

Stage 2: Repeat step 3 of the preparation stage with the remaining families.

Stage 3: Repeat steps 1 and 2 above. When there are two families remaining, complete the heuristic using Stage 2 of H2.

5. The Lagrangian Relaxation

We test our heuristics against a Lagrangian relaxation of Problem S. The subsets of constraints in (3) that belong to families i with $K_i \geq 2$ are removed. Let l_{ij} be the Lagrangian multiplier associated with item (i, j) in the instance of constraint (3) that is relaxed. The relaxed Problem LS follows.

Problem LS: A Lagrangian Relaxation of Problem S

$$\text{Maximize} \quad L + \sum_{\substack{(i,j) \in \boldsymbol{N} \times \boldsymbol{N} \\ i: K_i \geq 2}} l_{ij} \cdot \left(\sum_{t=1}^{M/k_{ij}} y_{ij}^{t} - 1 \right) \tag{32}$$

$$\text{Subject to} \quad \sum_{i=1}^{N} x_i^b \cdot S_i + \sum_{i-1}^{N} \sum_{j=1}^{N} y_{ij}^q \cdot P_{ij} + L \leq T; \quad \forall \ t = 1, \cdots, M \tag{33}$$

$$\sum_{t=1}^{M/K_i} x_i^t = 1; \qquad \forall\ i = 1, \cdots, N \tag{34}$$

$$\sum_{t=1}^{M/k_{ij}} y_{ij}^t = 1; \qquad \forall\ (i, j) \in \{1\} \times \boldsymbol{N} \tag{35}$$

$$\sum_{j=1}^{N} y_{ij}^t - N \cdot x_i^t = 0; \qquad \forall\ i = 1, \cdots, N; \forall\ t = 1, \cdots, K_i \tag{36}$$

$$x_i^t, y_{ij}^t \sim 0-1; \qquad \forall\ i, j = 1, \cdots, N; \forall\ t = 1, \cdots, M \tag{37}$$

$$L \geq 0 \tag{38}$$

We employ the subgradient method, as outlined by Fisher (1981) to solve Problem LS. Let $\boldsymbol{?}^s$ be the vector of l_{ij}'s in the s^{th} iteration of the subgradient procedure, with $\boldsymbol{?}^0 = \boldsymbol{0}$ and $\boldsymbol{?}^{s+1} = \boldsymbol{?}^s + t^s \cdot \left(\boldsymbol{C}\boldsymbol{x}^s - \boldsymbol{d}\right)$. The matrix $\boldsymbol{C}$ is the one formed by the removed rows of Problem S, and the column vector $\boldsymbol{d}$ is the resulting RHS $(\boldsymbol{b} = \boldsymbol{1})$. The objective function in (32) can be rewritten as (39) below. The row vector $\boldsymbol{x}^s$ represents the vector of values of the decision variables in (39) below that solve Problem LS in iteration s.

$$L + \sum_{\substack{(i,j) \in \boldsymbol{N} \times \boldsymbol{N} \\ i: K_i \geq 2}} l_{ij} \cdot \left(\sum_{t=1}^{M/k_{ij}} y_{ij}^t - 1 \right) + 0 \cdot \left(\sum_{\substack{(i,j) \in \boldsymbol{N} \times \boldsymbol{N} \\ i: K_i = 1}} \sum_{t=1}^{M/k_{ij}} y_{ij}^t + \sum_{\substack{i \in \boldsymbol{N} \\ i: K_i \geq 2}} \sum_{t=1}^{M/K_i} x_i^t \right) \tag{39}$$

The step size, t^s, is given by (40) below. The objective value at iteration s is represented by $Z^D\left(l^S\right)$. We obtain the lower bound Z^* on Z^D using the heuristic H1 above. We begin with $u^0 = 2$. Whenever Z^D has failed to decrease after three iterations, u^2 is divided by two. We stop the subgradient procedure when Z^D has failed to change up to the fifth decimal place after ten iterations.

$$t^s = \frac{u^s \cdot \left[Z^* - Z^D \cdot \left(l^S \right) \right]}{\left\| \boldsymbol{C}\boldsymbol{x}^s - \boldsymbol{d} \right\|^2} \tag{40}$$

6. Example

We illustrate multi-family scheduling solution procedure with an example. In this example, a facility produces five families, each with five items. The time unit is arbitrarily chosen to be one week. The data are generated from uniformly distributed parameters as follows:

Parameter	Distribution
Family setup time (weeks)	U(0.015, 0.025)
Family setup cost ($)	U(100, 500)
Item setup time (weeks)	U(0.0012, 0.018)
Item setup cost ($)	U(50, 150)
Item holding cost ($)	U(0.01, 1.25)
Item demand mean (units)	U(10, 500)
Item demand standard deviation (% of demand mean)	U(0.5, 0.85)
Item production rate (units/week)	U(10,000, 15,000)
Item service level	U(0.85, 0.9999)

The data are:

Family setup times: $\boldsymbol{S} = (0.0226 \quad 0.0227 \quad 0.0221 \quad 0.0156 \quad 0.0189)$

Family setup costs: $\boldsymbol{A} = (\$420 \quad \$261 \quad \$450 \quad \$350 \quad \$189)$

Item setup times:

$$\boldsymbol{s} = \begin{pmatrix} 0.0027 & 0.0063 & 0.0099 & 0.0125 & 0.0177 \\ 0.0125 & 0.0140 & 0.0034 & 0.0065 & 0.0055 \\ 0.0136 & 0.0113 & 0.0081 & 0.0158 & 0.0013 \\ 0.0090 & 0.0053 & 0.0167 & 0.0032 & 0.0108 \\ 0.0026 & 0.0086 & 0.0154 & 0.0134 & 0.0030 \end{pmatrix}$$

Item setup costs:

$$\boldsymbol{a} = \begin{pmatrix} \$122 & \$66 & \$133 & \$82 & \$67 \\ \$100 & \$85 & \$128 & \$109 & \$94 \\ \$94 & \$79 & \$57 & \$139 & \$95 \\ \$61 & \$73 & \$62 & \$77 & \$102 \\ \$103 & \$141 & \$111 & \$140 & \$146 \end{pmatrix}$$

Item holding cost:

$$\boldsymbol{h} = \begin{pmatrix} \$1.14 & \$0.19 & \$0.32 & \$1.13 & \$1.00 \\ \$0.25 & \$0.59 & \$0.93 & \$1.19 & \$0.07 \\ \$0.63 & \$0.47 & \$0.73 & \$0.66 & \$1.18 \\ \$1.07 & \$0.11 & \$0.90 & \$0.92 & \$0.44 \\ \$0.19 & \$0.73 & \$0.23 & \$1.01 & \$0.70 \end{pmatrix}$$

Item demand mean:

$$\boldsymbol{d} = \begin{pmatrix} 378 & 218 & 379 & 251 & 270 \\ 400 & 440 & 227 & 11 & 54 \\ 181 & 34 & 239 & 106 & 350 \\ 16 & 152 & 258 & 356 & 28 \\ 418 & 396 & 292 & 105 & 364 \end{pmatrix}$$

Item demand standard deviation:

$$\boldsymbol{s} = \begin{pmatrix} 262.45 & 128.77 & 222.81 & 182.08 & 214.25 \\ 312.76 & 340.74 & 175.86 & 9.01 & 28.05 \\ 94.55 & 28.68 & 157.19 & 87.31 & 184.63 \\ 8.59 & 82.79 & 210.61 & 193.77 & 17.83 \\ 255.90 & 221.40 & 236.26 & 53.00 & 276.09 \end{pmatrix}$$

Item production rate:

$$\boldsymbol{p} = \begin{pmatrix} 14{,}340 & 10{,}887 & 12{,}421 & 10{,}955 & 14{,}201 \\ 14{,}592 & 13{,}598 & 10{,}765 & 11{,}720 & 14{,}108 \\ 11{,}700 & 10{,}508 & 12{,}431 & 12{,}721 & 12{,}237 \\ 14{,}193 & 10{,}303 & 13{,}211 & 11{,}242 & 10{,}769 \\ 12{,}700 & 11{,}145 & 11{,}445 & 13{,}540 & 12{,}346 \end{pmatrix}$$

Item service level:

$$\boldsymbol{SL} = \begin{pmatrix} 0.9743 & 0.8881 & 0.9044 & 0.9673 & 0.9724 \\ 0.9398 & 0.9606 & 0.9321 & 0.9447 & 0.9723 \\ 0.8918 & 0.9177 & 0.8709 & 0.9133 & 0.8924 \\ 0.8927 & 0.8769 & 0.8689 & 0.9751 & 0.9988 \\ 0.9156 & 0.8712 & 0.9405 & 0.8663 & 0.9621 \end{pmatrix}$$

The multi-family algorithm in Karalli and Flowers (2004) is used to obtain the solution, $\left(T^{*}, K^{*}, k^{*}\right)$, given next.

Total average cost	TC=\$9,933.87
Basic period length	$T^{*}=0.511$
Family multipliers	$\boldsymbol{k}^{*}=\left(2,\ 2,\ 2,\ 1,\ 2\right)$
Item multipliers	$\boldsymbol{K}^{*}=\begin{bmatrix} 1 & 8 & 1 & 1 & 1 \\ 1 & 1 & 2 & 4 & 2 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 2 \\ 8 & 1 & 1 & 8 & 2 \end{bmatrix}$

The Lagrangian upper bound is 0.21616. Applying each of the three heuristics yields the results below

Heuristic	Objective Value	% Difference from Lagrangian Upper Bound
H1	0.21263	1.64
H2	0.21379	1.12
H3	0.20698	4.25

The three heuristics were applied to 250 randomly generated problems from the distribution of parameters given above. Results are tabulated below.

Difference from Lagrangian Upper Bound

Heuristic	Mean	Median	Mode	Standard Deviation
H1	1.9391%	0.0001%	0.0000%	4.4186%
H2	2.2948	0.0003	0.0000	4.8530
H3	8.7440	5.0769	0.0000	11.8914

Heuristic H1, which removed constraints and no variables, performed the best with a mean of 1.9391% difference from the Lagrangian relaxation compared to 2.2948% and 8.7440% for H2 and H3, respectively. Heuristic H2 has the added desirable feature of further reducing the MIP by removing variables in addition to constraints. The added computational efficiency raises the mean difference from the Lagrangian relaxation's objective value from 1.9391% to 2.2948%. Dividing the MIP into more sub-problems (H3) incurs additional costs in solution quality. For very large problems, the use of an approach like H3 could justify the additional penalty.

7. Summary

In this paper, we continue the study of the MFELSP—SS by taking the solution of the Family Planning Problem (FPP), $(\mathrm{T},\mathbf{K},\mathbf{k})$, and solving the Family Periodic Loading Problem, which actually schedules the family and its items across M periods. We offer three heuristic procedures that increasingly trade off speed for solution quality. Heuristic H1, the slowest, yielded objective values with a mean difference of 1.9391% from the Lagrangian relaxation. Heuristic H2, faster than H1, yielded objective values 2.2948 % lower than the Lagrangian upper bound. Heuristic H3, the fastest, yielded objective values 8.7440% lower than the Lagrangian upper bound.

One of the important deficiencies of basic period approaches that use multipliers is that there is no guarantee of a feasible schedule, especially when the equal lot size rule is enforced. In this paper, we kept the equal lot-size rule because of the existence of safety stocks, arguing that allowing the lot sizes to vary would further complicate the problem by requiring the safety stock levels to vary as well. These safety stock levels were computed in the FPP as part of the optimization problem. Taken together, the solution procedures developed for the FPP and the FPLP should allow practitioners to solve the multiple family economic lot scheduling problem with safety stocks more effectively than previously available methodologies. Environments where production is to stock, and not to customer order, where demand is uncertain, so safety stocks are required, and where stability and efficiency in production schedules are sought, should benefit from these procedures. Thus, the heuristics should be particularly effective in Just-In-Time manufacturing environments.

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