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The Multiple-Family ELSP with Safety Stocks

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Abstract

The Economic Lot Sizing Problem (ELSP) with normally distributed, time-stationary demand is considered in a manufacturing setting where the relevant costs include family setup costs, item setup costs, and inventory holding costs for both cycle and safety stocks. A family is a subset of the items that share a common family setup with its associated setup cost and setup time. Each item within the family may have its own setup time and setup cost. The families form a partition of the set of items manufactured on a single facility. The safety stock level for any item is a function of the time interval between production runs for the item, the service level specified, and the variance of its demand. We consider safety stocks explicitly in the formulation, as their holding costs vary nontrivially with the model's decision variables.

The Multiple-Family ELSP with safety stocks differs from multi-level inventory models with family setups in that the former assumes non-instantaneous inventory replenishment and considers the cost of holding safety stocks; the latter assumes instantaneous replenishment and does not directly assess the impact of safety stock levels on the total cost.

An efficient solution procedure is developed for this model. Properties of the non-convex feasible space are identified and used in the solution approach. The solution to the mathematical model is comprised of the basic period length, the family multipliers, and the item multipliers that give the lowest total cost of setups and carrying inventory. The family multipliers and items multipliers are restricted to integer powers of two.

1. Introduction

Despite the increasing trend toward adopting lean manufacturing strategies, many production processes continue to benefit economically from producing to stock and carrying substantial inventories. Despite "concerted efforts to reduce setup times in Just-In-Time and related programs," state Federgruen and Katalan (1996), setup times "remain significant in most practical production settings." A typical process of this nature consists of a single facility (or machine) that produces multiple items (or products) one unit at a time, requiring cost-intensive and/or time-intensive changeovers between products. Such processes are prevalent as it is generally "more economical to purchase one high speed machine that is capable of producing many products than to purchase many dedicated machines" (Dobson, 1987). Once produced, each unit is placed in inventory, where it begins to incur holding costs. We assume that each item has a time-stationary, normally distributed demand with known parameters. We further assume that each item has a unit inventory holding cost per unit time, and a known, constant production rate. Additional exogenous inputs to the problem are production rates and service level requirements.

1.1 Lego Block Example

An illustration of a production setting that fits the given description is the Lego block example adapted from Muckstadt and Roundy (see Graves et al., 1993). A facility manufactures Lego blocks in three different sizes, each in three different colors. Each Lego block is given an (i, j) designation, where i represents the size and j, the color. The sizes are small, medium, and large, represented by i=1, 2 and 3, respectively. The colors are red, blue, and green, represented by j=1, 2 and 3, respectively. We denote a small, red block by $(1,1)$, and use this notation for all other eight items. The facility may produce item $(1,1)$ in lot sizes of 100 units. Demand for each of the nine items is time-stationary and normally distributed with known parameters. The production rate for each item is known and constant. For item $(1,1)$ the mean demand is r_{11}, the standard deviation of demand is σ_{11}, and the production rate is p_{11}.

Suppose that production of $(1,1)$ blocks is complete, and we wish to produce $(1,2)$ blocks, next. We need to clean the small mold for a change of paint color. This is the item setup. Each item has a setup cost and a setup time. The setup cost and setup time for item $(1,1)$ are a_{11} and s_{11}, respectively. Switching production from $(1,1)$ to $(2,3)$ requires both a family setup, replacing the small mold with the medium mold, and an item setup. Each family has a setup cost and a setup time. The setup cost for family 2 is A_2, the setup time, S_2. Setup times consume part of the facility's total capacity. The holding cost for item $(1,1)$ is h_{11}. The required service level for Lego block $(1,1)$ is SL_{11}.

The relevant costs in the MFELSP-SS are the sum of the setup costs and the inventory holding costs. We seek to minimize the average total average cost per unit time by seeking a balance between the setup frequencies of each family and their items, and the average working and safety stock levels of each of the nine block types.

Such a balance can be represented by a basic period length, T, family multipliers K_i, and item multipliers $k_{ij}, \forall\ i, j = 1,2,3$. The interval between setups for family i is $T \cdot K_i$, and $T \cdot K_i \cdot k_{ij}$ between setups for item (i, j). The expected average working stock level is

$$\frac{1}{2} r_{ij} \cdot \left(1 - \frac{r_{ij}}{p_{ij}}\right) \cdot T \cdot K_i \cdot k_{ij}. \tag{1}$$

With Z_{ij} as the standard normal deviate, the safety stock required to meet the service level, SL_{ij}, is

$$Z_{ij} \cdot \sigma_{ij} \cdot \sqrt{T \cdot K_i \cdot k_{ij}}. \tag{2}$$

The average family setup cost per unit time of family i is

$$\frac{A_i}{T \cdot K_i}. \tag{3}$$

The average item setup cost per unit time of item (i, j) is

$$\frac{a_{ij}}{T \cdot K_i \cdot k_{ij}}. \tag{4}$$

The average item working stock holding cost per unit time of item (i, j) is

$$\frac{1}{2} h_{ij} \cdot r_{ij} \cdot \left(1 - \frac{r_{ij}}{p_{ij}}\right) \cdot T \cdot K_i \cdot k_{ij}. \tag{5}$$

The average item safety stock holding cost per unit time of item (i, j) is

$$h_{ij} \cdot Z_{ij} \cdot \sigma_{ij} \cdot \sqrt{T \cdot K_i \cdot k_{ij}}. \tag{6}$$

If **K** is the set of K_i values and **k** is the set of k_{ij} values, the average total average cost per unit time for the entire process is given by (7) below.

$$C(T, \mathbf{K}, \mathbf{k}) = \sum_{i=1}^{3} \left(\frac{A_i}{TK_i} + \sum_{j=1}^{3} \left[\frac{a_{ij}}{TK_i k_{ij}} + \frac{1}{2} h_{ij} r_{ij} \left(1 - \frac{r_{ij}}{p_{ij}}\right) TK_i k_{ij} + h_{ij} Z_{ij} \sigma_{ij} \sqrt{TK_i k_{ij}} \right] \right). \tag{7}$$

We illustrate a cyclic schedule using the Lego block venue. Suppose that for the Lego block problem a basic period of length T was computed along with the following multiplier values:

Family	i	K_i	k_{ij}		
S	1	1	1	2	2
M	2	2	2	1	2
L	3	2	2	2	1
		j	1	2	3
		Item	R	B	G

A solution Example to the Lego Block Problem

A cycle of M = 4 consecutive basic periods will allow us to schedule the families and the items as indicated in the Table 1 above. Family 1, with $K_1 = 1$, will be scheduled every period. Families 2 and 3, with $K_2 = 2$ and $K_3 = 2$, will be scheduled every other period. Small red blocks, with $k_{11} = 1$, will be scheduled for production every time family 1 is scheduled, whereas small blue blocks and small green blocks, with $k_{12} = k_{13} = 2$, will be scheduled every other time family 1 is scheduled. Obtaining the solution, $(\mathrm{T}, \mathbf{K}, \mathbf{k})$, which comprises the basic period length, T, the vector $\mathbf{K}$ of family multiplier values, $\mathbf{K} = (K_1, K_2, K_3)$, and the matrix $\mathbf{k}$ of item multiplier values,

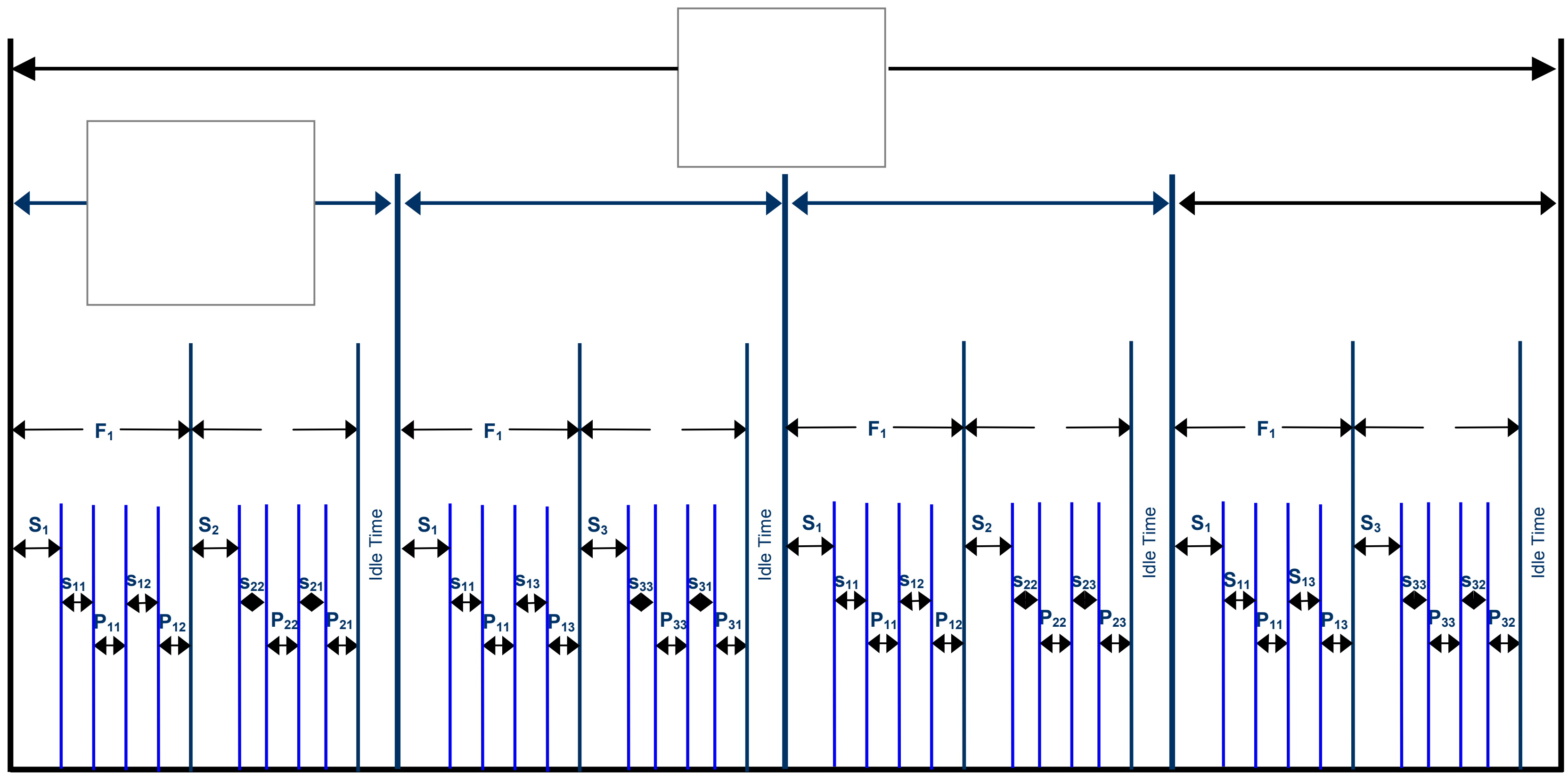
$$\mathbf{k} = \begin{bmatrix} k_{11} & k_{12} & k_{12} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix},$$

is the subject of this paper. The solution is used to create a cyclic schedule. An example of a cyclic schedule generated from the example solution given above is presented in Figure 1 below.

1.2 Literature Review

The extensively researched economic lot-scheduling problem (ELSP) addresses the deterministic, single-setup-stage case of the foregoing process. Gallego and Shaw (1997) show that the ELSP is strongly NP-hard under General Cyclic Schedules, Zero-Inventory Cyclic Schedules, Time-Invariant Cyclic Schedules, Lot-Invariant Cyclic Schedules, and Basic Period Cyclic Schedules. The last result is pertinent to this paper as the sought solution is in the form of a Basic Period Cyclic Schedule. Elmaghraby (1978) provides a review of the deterministic ELSP. The conventional ELSP faces a single machine, producing N items, each with known, constant demand and production rates. Each item has a unit inventory holding cost per unit time, and a unit backorder cost. Each item has a setup cost and setup time, both of which are sequence-independent. The relevant costs include setup costs, inventory holding costs, and backordering costs. The stochastic ELSP (SELSP) considers uncertain demand. Sox et al. (1999) provide a review the SELSP.

The Multiple-Family Economic Lot—Scheduling Problem with Safety Stocks (MFELSP—SS) extends the SELSP to multiple families, each having multiple items. Significantly costly and time-consuming changeovers occur in two stages. We will refer to the first stage as a family setup, and the second, an item setup. When the facility is setup for a particular family, only item setups are required when changeover takes place to another item in that family. Both setup stages are required for a changeover to an item in a different family. Without loss of generalization, we assume that each family, as well as each item, has a setup time and a setup cost.



F1
S1
s11
P11
s12
P12
S2
s22
P22
s21
P21
Idle Time
S3
s13
P13
s33
P33
s31
P31
s23
P23
s32
P32

Since Maxwell's seminal paper (1964), cycle-schedule solutions to the deterministic ELSP became standard. A cyclic schedule is comprised of multiple equal-length basic periods, executed repeatedly. Within a cycle, each item's production run is scheduled equal intervals apart. The length of these intervals is the product of a basic period length and an integer multiplier. The objective was to generate a cyclic schedule that minimizes the sum of the setup costs and inventory holding costs per unit time. See Madigan (1968), Hodgson (1970), Doll and Whybark (1973), Elmaghraby (1978), Graves (1980), Goyal (1984), Dobson (1987), and Axsäter (1987) for heuristic solution approaches. Grznar and Riggle (1997) provide the first ELSP optimal algorithm.

For reasons of practical implementation, restricting the intervals between each item's production run to the basic period length times a positive integer-power-of-two multiplier gained popularity after Roundy's critical paper (1988). Roundy shows that using integer-power-of-two multipliers, obtained by his polynomial-time Roundoff Algorithm (a heuristic), results in a total cost that is at most 6% higher than the lowest cost of the continuous relaxation of the ELSP. For research focusing on cyclic schedules with multipliers restricted to positive integer-powers-of-two, see Maxwell and Singh (1983), Dobson (1987), Roundy (1989), and Bourland and Yano (1997). In addition to using positive integer-power-of-two multipliers, Gallego and Roundy (1992) extend the ELSP to include finite backorder costs. Cyclic sequencing policies are suitable to the SELSP when production is coordinated with other scheduled activities, such as the procurement of raw materials (Gallego, 1990).

We adopt the approach to creating cyclic sequencing policies for the SELSP, which involves approximating the problem with a deterministic-demand model to obtain the multipliers and the basic period length for a cyclic schedule. The multipliers and the basic period form the basis for the subsequent step of creating the cyclic schedule, such as the example given in Figure 1 above. Dobson (1987) provides a procedure for creating such a schedule for the ELSP. A control rule is required for the stochastic problem to follow this schedule while responding to the stochastic inventory levels at each decision epoch. Graves (1980), Leachman and Gascon (1988), Gallego (1990), Leachman et al. (1991), Bourland and Yano (1994), and Federgruen and Katalan (1996) develop control rules using a deterministic cyclic schedule as the starting point.

R. G. Brown (1967) studied the ELSP with sequence independent setup times and an additional family setup time, common to all items. His procedure obtains item multipliers, and the basic period length, which minimize an EOQ-like total cost function. The next step in his solution is to obtain the economic lot sizes (ELS) from the multipliers and the basic period, which he uses, along with the basic period length, to create a cyclic schedule. Brown's treatment of the single—family model does not lend itself to the multiple—family case. Nonas and Thorstenson (2000) combine the multiple-family ELSP with the cutting stock problem. The setup cost is identical for each family. They provide a mathematical formulation but no solution.

Harris' (1913) key insight of the tradeoff between the frequency of ordering (corresponding to changeovers in the ELSP) and lot sizes—affecting average inventory levels—is germane to the ELSP. Higher changeovers frequencies result in increases in total fixed costs, and reductions in average inventory levels and carrying costs. Conversely, higher lot sizes result in reductions in changeover frequencies and total fixed costs, but raise average inventory levels and carrying costs. In a stochastic demand setting, safety stock for each item hedges against unexpected demand. The safety stock level for any item is an increasing function of the time interval between production runs for the item, the service level specified, and the variance of its demand. Since safety stock levels also play into the cost tradeoffs, we consider them explicitly in the formulation of the MFELSP—SS mixed integer-NLP.

Gallego (1990) uses a deterministic cyclic schedule, generated using mean demand rates, consisting of integer-power-of two multipliers according to Roundy's Roundoff algorithm (1988) and Dobson's periodic loading solution (1987). He then develops a recovery cyclic schedule to restore the inventory levels to their required starting points after disruptions due to random demand. Gallego computes safety stocks after generating the deterministic cyclic schedule. Lower long-run average costs can be obtained by explicitly considering the cost of holding safety stocks in the formulation used to generate the multipliers for the cyclic schedules. Furthermore, the safety stocks assume that the times between each item's production run are fixed, when, in fact, they vary due to demand variation. Bourland and Yano are the first to consider safety stock explicitly in the formulation of the SELSP with a fixed sequence schedule. Their approach allows an item's lot to be run ahead of schedule if its inventory unexpectedly runs out. The current item in production is relegated to overtime production, where a linear cost with respect to time is assessed. The model computes the safety stock for each item to balance setup costs, holding costs of working stock and safety stock, and overtime costs. Safety stock, in Bourland and Yano's model does not so much address a service level as it does the use of overtime. In this paper, we explicitly consider safety stock in our formulation of the MFELSP—SS.

A special case of the multi-item, multi-stage production system, the family model, is due to Muckstadt and Roundy (see Graves, 1993). This model considers family setups in a deterministic demand setting with instantaneous replenishment. The restriction of a single facility that can only produce one item at a time is not part of their model. Muckstadt and Roundy's model has a cost penalty of at most 6%, in contrast to the 44% maximum penalty for the general case of the multi-item, multi-stage production. Federgruen, Groenevelt, and Tijms (1984) examine a stochastic multi-family, multi-item variant of the joint replenishment problem (JRP) with compound Poisson demand processes.

Our model differs from Muckstadt and Roundy's multi-level inventory models with family setups in that the MFELSP assumes non-instantaneous inventory replenishment and considers the cost of holding safety stocks. The multi-item, multi-stage family model assumes instantaneous replenishment and does not directly assess the impact of safety stock levels on the total cost. As such there is no production capacity to consider. Considering safety stock costs, we shall see, add a term to the objective function, complicating the solution procedure of the model. The presence of the production capacity in our model adds a feasibility constraint, which also requires a solution procedure different from that of Muckstadt and Roundy (1993). We provide useful properties of the solution to our problem, which facilitates its search.

In §2 we formulate Problem P of the MFELSP-SS. In §3 we derive useful solution properties as well as mathematical properties of the mathematical problem. In §4, we present a Problem R, a continuous relaxation of Problem P. We develop a functional approximation of

Problem R in §5. We present an efficient algorithm in §6. In §7, we illustrate our procedure with a numerical example and provide a summary in §8.

2. Problem Formulation

In this section, we develop the MFELSP—SS. We seek to minimize the total average relevant cost per unit time in a cyclic schedule. The relevant costs include family setup costs, item setup costs, and inventory holding costs for both cycle and safety stocks. We consider a Type I service criterion on each item. The differentiation of service levels across items is desirable as Brown (1967) observes that, "with remarkable regularity," the mean demand rates of products within multi-item facilities tend to be lognormally distributed. In other words, a small percentage of the items generate a high percentage of the total revenue. A high service level may be required for items that generate the bulk of the firm's revenues. Low-revenue generating items may be given lower service level requirements.

Problem Environment:

The MFELSP—SS is a continuous-time, infinite-horizon extension of the ELSP where

N families, each having N items, are produced in the same facility, one unit at a time.
For each family, there is a

- sequence-independent setup cost
- sequence-independent setup time

For each item, the demand is

- time-stationary,
- normally distributed, with
- known mean and standard deviation,
- uncorrelated with other items,
- not substitutable with other items.

For each item,

- there is a known, constant production rate,
- there is a sequence-independent setup cost,
- there is a sequence-independent setup time,
- there is a specified Type I service level,
- safety stock is maintained order to meet the specified service levels, and
- there is no backlogging.

For notational convenience, we take N to be both the number of families and items within each family, without loss of generality (w.l.o.g.). When the number N exceeds the actual number of items in any family, or exceeds the number of families, we simply create dummy items and/or families and assign the value of zero to their parameters.

The following parameters are inputs to the problem:

S_i	Setup time for family i.
s_{ij}	Setup time for the j^{th} item in family i.
A_i	Setup cost for family i.
a_{ij}	Setup cost for item j in family i.
d_{ij}	Demand mean for item j in family i.
σ_{ij}^2	Demand variance for item j in family i.
p_{ij}	Production rate for item j in family i.
ρ_{ij}	$= d_{ij} / p_{ij}$, production capacity needed by item j of family i
ρ	$= \Sigma_{i\in N}\Sigma_{j\in N} (d_{ij} / p_{ij})$, Total production capacity required
h_{ij}	Inventory carry cost for item j in family i.
Z_{ij}	Standard deviation from N(0,1) corresponding to the service level required for item j in family i.

The decision variables of the problem follow:

T	Length of Basic Period
K_i	Multiplier of family i
k_{ij}	Multiplier for item j of family i

The following notation will appear in the discussion:

$C(\cdot)$	Cost Function Evaluated at $(\cdot)$
$G(\cdot)$	Feasibility constraint, see formulation below
T	Length of Basic Period
τ	Length of Cycle Time
N	Set of subscripts of i or j from 1 to N.
K	N-Vector of K_i's
k	(N x N)-matrix of k_{ij}'s
$\mathbf{k}_{i\bullet}$	The vector of all the item multipliers in family i.
P	$=\{2^p: p \in Z_+\}$, set of integer-powers-of-two

Ideally, the firm would like to adopt an inventory policy that

- minimizes the total relevant operating costs,
- is feasible to implement given the facility's existing capabilities—production rates and capacity, and
- complies with its service criteria.

Problem P:

We represent the solution to Problem P by (T, **K**, **k**), where T, **K**, and **k** are defined above. The objective is to find the cycle time, T, and the multipliers K_i and k_{ij} for each family i and its member item j [i.e., to find (T, **K**, **k**)] so as to

$$\text{Minimize} \quad C(T,\mathbf{K},\mathbf{k}) = \sum_{i\in\mathbf{N}} \frac{A_i}{TK_i} + \sum_{i\in\mathbf{N}} \sum_{j\in\mathbf{N}} \left(\frac{a_{ij}}{TK_ik_{ij}} + b_{ij}TK_ik_{ij} + g_{ij}\sqrt{TK_ik_{ij}} \right) \tag{8}$$

Subject to $G(T,\mathbf{K},\mathbf{k}) = \sum_{i\in\mathbf{N}} \frac{S_i}{K_i} + \sum_{i\in\mathbf{N}} \sum_{j\in\mathbf{N}} \frac{s_{ij}}{K_i k_{ij}} - (1-\rho)\cdot T \leq 0$ (9)

$T>0$ (10)

$K_i \in \mathbf{P} \qquad \forall\, i \in \mathbf{N}$ (11)

$k_{ij} \in \mathbf{P} \qquad \forall\, (i,j) \in \mathbf{N}\times\mathbf{N}$ (12)

Where $b_{ij} = \frac{1}{2} h_{ij} r_{ij} (1-\rho_{ij})$ (13)

$g_{ij} = h_{ij} Z_{ij} \sigma_{ij}$ (14)

The objective function, (1), is an average total cost per unit time of all family and item setups, and of holding working and safety stocks. Constraint (2) is a necessary feasibility condition. Constraint (3) can be relaxed to a weak inequality without loss of accuracy. Constraints (4) and (5) restrict the family and item multipliers, respectively, to integer-powers-of-two.

3. Properties

3.1 Solution Properties

The following properties are useful in our search for a solution to Problem P:

Proposition 3.1: Let $(T^*, \mathbf{K}^*, \mathbf{k}^*)$ solve Problem P with $\mathbf{K}^* = (K_1^*, \cdots, K_N^*)$, such that w.l.o.g. $K_1^* \leq K_2^* \leq \cdots \leq K_N^*$. Then, for T' and $\mathbf{K}'$, with $T' = T^* K_1^*$, and $\mathbf{K}' = (1, K_2^*/K_1^*, \cdots, K_N^*/K_1^*)$, then $(T', \mathbf{K}', \mathbf{k}^*)$ is optimal for Problem P.

Proof of Proposition 3.1: The objective value C^* =

$$C(T^*, \mathbf{K}^*, \mathbf{k}^*) = \sum_{i\in\mathbf{N}} \frac{A_i}{T^* K_i^*} + \sum_{i\in\mathbf{N}} \sum_{j\in\mathbf{N}} \left(\frac{a_{ij}}{T^* K_i^* k_{ij}^*} + b_{ij} T^* K_i^* k_{ij}^* + g_{ij}\sqrt{T^* K_i^* k_{ij}^*} \right) \tag{15}$$

Set $T' = T^* K_1^*$. Now,

$$C\left(T^*,\mathbf{K}^*,\mathbf{k}^*\right) = \sum_{i\in\mathbf{N}} \frac{A_i}{T'\frac{K_i^*}{K_1^*}} + \sum_{i\in\mathbf{N}}\sum_{j\in\mathbf{N}}\left(\frac{a_{ij}}{T'\frac{K_i^*}{K_1^*}k_{ij}^*} + b_{ij}T'\frac{K_i^*}{K_1^*}k_{ij}^* + g_{ij}\sqrt{T'\frac{K_i^*}{K_1^*}k_{ij}^*}\right)$$

$$= \sum_{i\in\mathbf{N}} \frac{A_i}{T'K'_i} + \sum_{i\in\mathbf{N}}\sum_{j\in\mathbf{N}}\left(\frac{a_{ij}}{T'K'_i k_{ij}^*} + b_{ij}T'K_i{}'k_{ij}^* + g_{i,j}\sqrt{T'K'_i k_{ij}^*}\right) \tag{16}$$

$$= C\left(T',\mathbf{K'},\mathbf{k}^*\right).$$

QED.

Proposition 3.2: Let $\left(T^*,\mathbf{K}^*,\mathbf{k}^*\right)$ solve Problem P with $\mathbf{k}_{i\bullet}^* = \left(k_{i1}^*,\cdots,k_{iN}^*\right)$, for some $i \in N$. Moreover, and w.l.o.g., let $k_{i1}^* \le k_{i2}^* \le \cdots \le k_{iN}^*$. Then $k_{i1}^* = 1$.

Proof of Proposition 3.2: A proof by contradiction is employed with the assumption that $k_{m1}^* \ne 1$, i.e. $k_{m1}^* \ge 2$, for some $m \in \mathbf{N}$ results in the contradiction that the solution is not optimal. The objective value C^* =

$$\begin{aligned} C\left(T^*,\mathbf{K}^*,\mathbf{k}^*\right) &= \sum_{\substack{i\in\mathbf{N}\\ i\ne m}} \frac{A_i}{T^*K_i^*} + \sum_{\substack{i\in\mathbf{N}\\ i\ne m}}\sum_{j\in\mathbf{N}}\left(\frac{a_{ij}}{T^*K_i^*k_{ij}^*} + b_{ij}T^*K_i^*k_{ij}^* + g_{ij}\sqrt{T^*K_i^*k_{ij}^*}\right) \\ &+ \frac{A_m}{T^*K_m^*} + \frac{a_{m1}}{T^*K_m^*k_{m1}^*} + b_{m1}T^*K_m^*k_{m1}^* + g_{m1}\sqrt{T^*K_m^*k_{m1}^*} \\ &= \sum_{\substack{j\in\mathbf{N}\\ j\ne 1}}\left(\frac{a_{ij}}{T^*K_m^*k_{mj}^*} + b_{mj}T^*K_m^*k_{mj}^* + g_{mj}\sqrt{T^*K_m^*k_{mj}^*}\right). \end{aligned} \tag{17}$$

Let $K'_m = K_m^*k_{m1}^*$, and $k'_{m1} = 1$. Substituting K'_m for K_m^*, and k'_{m1} for k_{m1}^* in (17), all the terms in (17) remain identical in value except for $A_m\Big/\left(T^*K'_m\right)$. In fact, $\frac{A_m}{T^*K'_m} < \frac{A_m}{T^*K_m^*}$

because $K'_m = K^*_m k^*_{m1} > K^*_m$. The desired contradiction is reached: $C(T^*, \mathbf{K}^*, \mathbf{k}^*)$ is not optimal. QED.

Propositions 3.1 and 3.2 above yield the convenient property that an optimal solution always exists where the smallest family multiplier is equal to one, and for each family the smallest item multiplier is equal to one. A solution in the form just describe will be henceforth referred to as a solution in standard anchor form, or AF.

Definition 3.3: Let $\left(T^*, \mathbf{K}^*, \mathbf{k}^*\right)$ solve Problem P. Then $\left(T^*, \mathbf{K}^*, \mathbf{k}^*\right)$ is in anchor form (AF) if the following conditions hold:

- $0 < T^* \in \mathbb{R}$,
- $\mathbf{K}^* = (1, K_2, \cdots, K_N)$, with $1 \le K_2 \le \cdots \le K_N$,
- $\forall\ i \in \mathbf{N}, K_i \in \mathbf{P}$,
- $\mathbf{k}^*$ is an $(\mathbf{N} \times \mathbf{N})$ – matrix whose i^{th} row, $\mathbf{k}_{i\bullet}$, is the vector of item multipliers for family i,
- $\mathbf{k}_{i\bullet} = (1, k_{i2}, \cdots, k_{iN})$, with $1 \le k_{i2} \le \cdots \le k_{iN}$,
- $\forall\ (i,j) \in \mathbf{N} \times \mathbf{N}, k_{ij} \in \mathbf{P}$.

3.2 Function Properties

We observe that $C(T, \mathbf{K}, \mathbf{k})$ is not convex in T as the safety-stock cost terms are polynomials to the degree ½ and therefore concave. An instance of the objective function with N = 1, a = 1, b = 1, and g = 1 is $C(T) = \frac{1}{T} + T + \sqrt{T}$. It is graphed below in Figure 2. Similarly, $C(T, \mathbf{K}, \mathbf{k})$ is not convex in K_i, $\forall\ i \in \mathbf{N}$. Nor is the function convex in $k_{ij}, \forall (i,j) \in \mathbf{N} \times \mathbf{N}$.

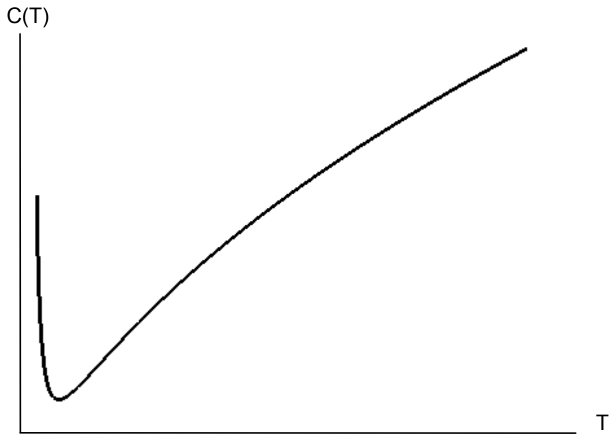


Figure 2: Graph of $C(T) = 1/T + T + T^{1/2}$

The following lemma will be used in the solution procedure.

Lemma 3.5: C(T, **K**, **k**) is monotone decreasing in T until it reaches a minimum point at T^* after which it is monotone increasing.

Proof of Lemma 3.5: Fix **K** and **k** without loss of generalization, and let

$$\bar{A} = \sum_{i \in \mathbf{N}} \frac{A_i}{K_i} + \sum_{i \in \mathbf{N}} \sum_{j \in \mathbf{N}} \frac{a_{ij}}{K_i \cdot k_{ij}}, \tag{18}$$

$$\bar{b} = \sum_{i \in \mathbf{N}} \sum_{j \in \mathbf{N}} b_{ij} \cdot K_i \cdot k_{ij}, \tag{19}$$

and

$$\bar{g} = \sum_{i \in \mathbf{N}} \sum_{j \in \mathbf{N}} g_{ij} \cdot \sqrt{K_i \cdot k_{ij}}. \tag{20}$$

Then,

$$C(T, \mathbf{K}, \mathbf{k}) = \frac{\bar{A}}{T} + \bar{b} \cdot T + \bar{g}\sqrt{T}. \tag{21}$$

The F.O.C. is

$$-\frac{\bar{A}}{T^2} + \bar{b} + \frac{\bar{g}}{2\sqrt{T}} = 0. \tag{22}$$

For T = ϵ, where $\epsilon > 0$ is appropriately small, the LHS of equation (15) is negative but always increasing in T until the RHS of (15) becomes equal to zero and then becomes positive thereafter. This means that equation (14) is first decreasing in T until it reaches a minimum point and then becomes increasing in T. T always exists and (15) is decreasing for $T \in (0, [8\bar{A}/\bar{g}]^{2/3})$ and increasing thereafter. QED.

4. A Continuous Relaxation

The properties of Problem P described in §4 above allow for a restricted search for a solution in AF. The following relaxation of Problem P, *Problem R*, and its derived properties will be used to develop an algorithm similar to Roundy's Roundoff Algorithm (1988), but modified to solve Problem P.

Problem R:

In the continuous relaxation that follows, Problem R will be separable in its variables.

Variable Substitutions

$Y_i = TK_i, \forall\ i \in \mathbf{N}$.

$X_{ij} = TK_i k_{ij}, \forall\ (i,j) \in \mathbf{N} \times \mathbf{N}$.

$\mathbf{Y}$ = N-vector of Y_i's.

$\mathbf{X}$ = $N \times N$ matrix of X_{ij}'s.

Find ($\mathbf{X}$, $\mathbf{Y}$) so as to:

$$\text{Minimize} \quad C(\mathbf{X},\mathbf{Y}) = \sum_{i\in\mathbf{N}} \frac{A_i}{Y_i} + \sum_{i\in\mathbf{N}} \sum_{j\in\mathbf{N}} \left(\frac{a_{ij}}{X_{ij}} + b_{ij} \cdot X_{ij} + g_{ij}\sqrt{X_{ij}} \right) \tag{23}$$

$$\text{Subject to} \quad G(\mathbf{X},\mathbf{Y}) = \sum_{i\in\mathbf{N}} \frac{S_i}{Y_i} + \sum_{i\in\mathbf{N}} \sum_{j\in\mathbf{N}} \frac{s_{ij}}{X_{ij}} - (1-\rho) \le 0 \tag{24}$$

$$Y_i - X_{ij} \le 0; \qquad \forall\ i \in \mathbf{N}, j \in \mathbf{N} \tag{25}$$

$$\mathbf{X},\ \mathbf{Y} \ge 0. \tag{26}$$

KKT Conditions for Problem R:

$\exists u \in \mathbf{R}, v_{ij} \in \mathbf{R}, \forall (i,j) \in \mathbf{N} \times \mathbf{N}$, such that all constraints hold and

$$u \ge 0,\ v_{ij} \ge 0, \qquad \forall\ i \in \mathbf{N}, j \in \mathbf{N} \tag{27}$$

Complementarity Conditions:

$$u \cdot G(\mathbf{X},\mathbf{Y}) = 0 \tag{28}$$

$$v_{ij} \cdot \left(Y_i - X_{ij}\right) = 0 \qquad \forall\ i \in \mathbf{N}, \forall\ j \in \mathbf{N} \tag{29}$$

Gradient Conditions:

$$-\frac{A_i}{Y_i^2} - u \cdot \frac{S_i}{Y_i^2} + \sum_{j \in \mathbf{N}} v_{ij} = 0 \qquad \forall\ i \in \mathbf{N} \tag{30}$$

$$-\frac{a_{ij}}{X_{ij}^2} - u \cdot \frac{s_{ij}}{X_{ij}^2} + b_{ij} + \frac{1}{2}\frac{g_{ij}}{\sqrt{X_{ij}}} - v_{ij} = 0 \qquad \forall\ (i,j) \in \mathbf{N} \times \mathbf{N} \tag{31}$$

Proposition 4. 1: Let $(\mathbf{X}, \mathbf{Y})$ be a KKT point to an instance of Problem R**.** Furthermore let **N** be the ordered set of indices so that $\forall$ i $\in$ **N** ($\forall$ j $\in$ **N**), $Y_i \le Y_{i+1}$ $\left(X_{ij} \le X_{i,j+1}\right)$. For each family i $\in$ **N**, (a) $\exists$ j $\in$ **N** such that $X_{ij} = Y_i$, and (b) $\forall$ j $\in$ **N**, $X_{ij} \ge Y_i$.

Proof of Proposition 4. 1: By constraint (18) of Problem R, (b) is true, i.e.

$\forall\ i \in \mathbf{N}, \nexists\ j \in \mathbf{N}, \ni X_{ij} < Y_i$.

To prove (a) is true we proceed by a proof by contradiction. Since we assume (a) is not true, we would have

$$\sum_{j \in \mathbf{N}} v_{ij} = 0. \tag{32}$$

Now, Consider the KKT condition (23). The result in (25) would preclude (23) from having a real solution. QED.

5. A Function Approximation

The basic structure of the MFELSP-SS cost function is of the form

$$\frac{A}{x} + bx + g\sqrt{x} \tag{33}$$

Expression (26) cannot be minimized by using a closed form solution. The obviation of numerical searches is especially appealing as the number of families and items increase. The following is used to approximate expression (26)

$$\frac{A}{x} + b \cdot x + \beta_X \cdot g \cdot \log(x). \tag{34}$$

Expression (27) is minimized when

$$-\frac{A}{x^2}+b+\frac{\beta_X g}{x}=0. \tag{35}$$

The quadratic equation yields

$$x=\frac{-\beta_X g\pm\sqrt{(\beta_X g)^2+4Ab}}{2b} \tag{36}$$

Proposition 5.1: Expression (27) is minimized with the largest value of x in equation (29), and the solution always exists.

Proof of Proposition 5.1: Noting that A, b, $\beta_X > 0$ and that (26), which (27) proxies, is defined on the interval (0, ∞), only the positive value of x in equation (36) will serve as a solution. It exists because $\sqrt{(\beta_X g)^2+4Ab} > \sqrt{(\beta_X g)^2}=\beta_X g$. QED.

The term $\beta_X \log(x)$ is substituted for $\sqrt{x}$, noting that β_X is dependent on x by

$$\beta_X=\frac{\sqrt{x}}{\log(x)}. \tag{37}$$

Using ordinary least squares estimates of the regression function $\sqrt{x}=\beta_0+\beta_X \log(x)$, we obtain β_X for ranges of values of x and display them in Table 1 below.

x-Interval	β_0	β_x
(0-1]	0.8830	0.2146
(1-2]	0.9869	0.6009
(2-4]	0.8066	0.8499
(4-8]	0.3489	1.1770
(8-16]	-0.7445	1.7002
(16-32]	-2.7196	2.4045
(32-64]	-6.2031	3.4005
(64-128]	-12.1067	4.8092
(128-256]	-21.8356	6.8012

Table 1: Regression Estimators over Various Ranges of Values of x

Since the β_0 term disappears upon taking the first derivative, only β_x is of interest. Figure 2 below displays the piecewise linear estimated curve superimposed over the function $y = \sqrt{x}$.

The following lemma is used to develop the search procedure for x.

Lemma 5.2: x is a decreasing function of β_X in (36). Moreover, x decreases at a decreasing rate in (36).

Proof of Lemma 5.2: The first derivative of (36) is negative. The second derivative is positive. The first derivative is given by (38) below. Condition (39) is always true because setup costs and holding costs are always positive.

$$\frac{dx}{d\beta_X} = \frac{-g + \left[(\beta_X g)^2 + 4Ab\right]^{-\frac{1}{2}} \left(\beta_X g^2\right)}{2b} < 0 \tag{38}$$

because

$$4Ab > 0 \tag{39}$$

Condition (39) also assures that the second derivative of (36) is positive.

Function Approximation Solution Procedure

To solve (36) use Table 1 as follows:

- Compute x using (40) below for each of the values of β_X listed in Table 1 above.

$$x = \frac{-\beta_X g + \sqrt{(\beta_X g)^2 + 4Ab}}{2b} \tag{40}$$

- Select the computed value of x corresponding to the β_X where x is in the interval (in the first column of Table 1). This corresponds to Figure 3 below.
- Alternatively, there might be two adjacent rows where the computed values of x corresponding to the β_X in one row is in the interval of the β_X in the other row, and vice versa. Such an occurrence is depicted in Figure 4 below. For the purposes of rounding off to an integer-power-of-two, as we will do in §6, taking an average of the two values of x is sufficient.

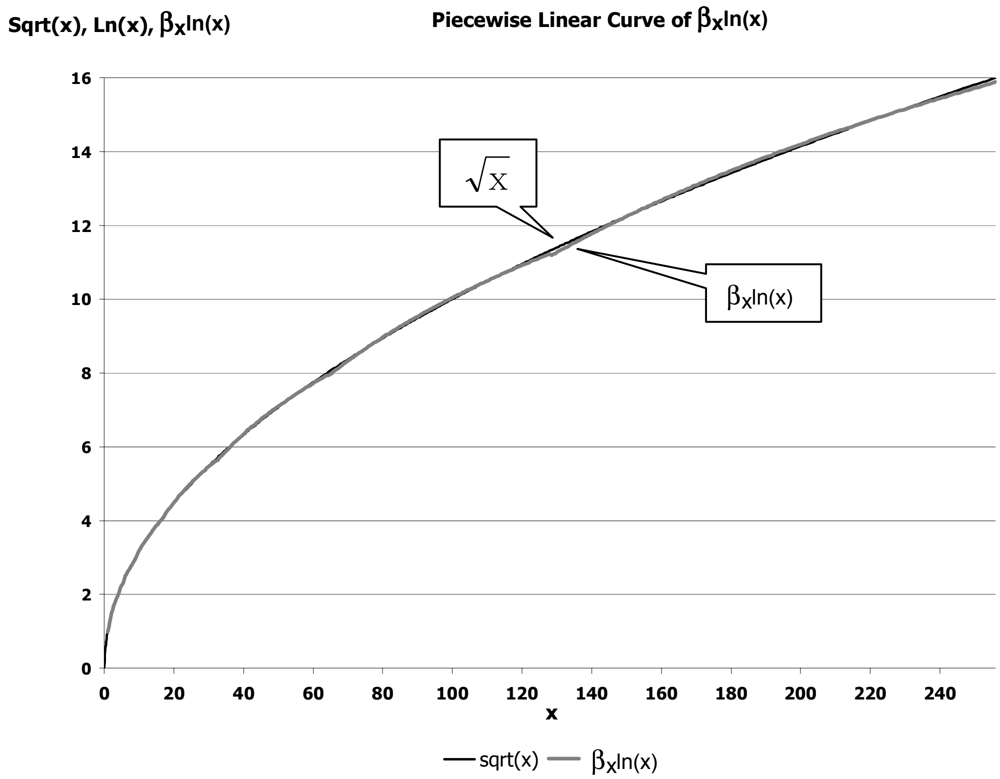


Figure 2: Piecewise Linear Curve of $\beta_X\ln(x)$.

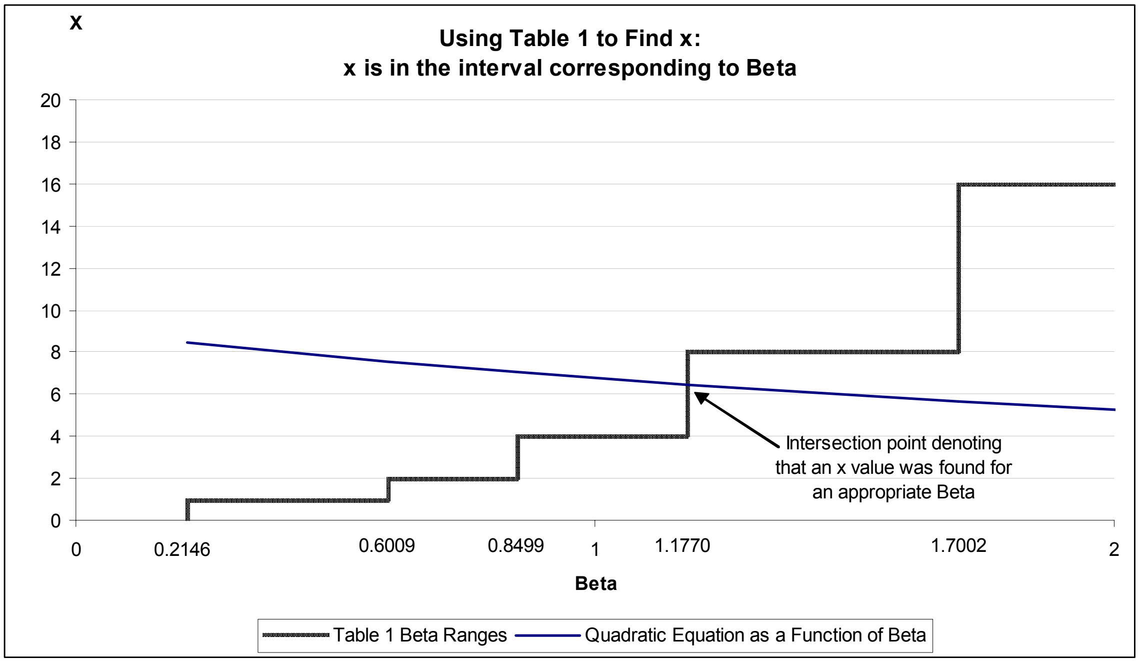


Figure 3: Using Table 1 to Find x: x is in the interval corresponding to β_x .

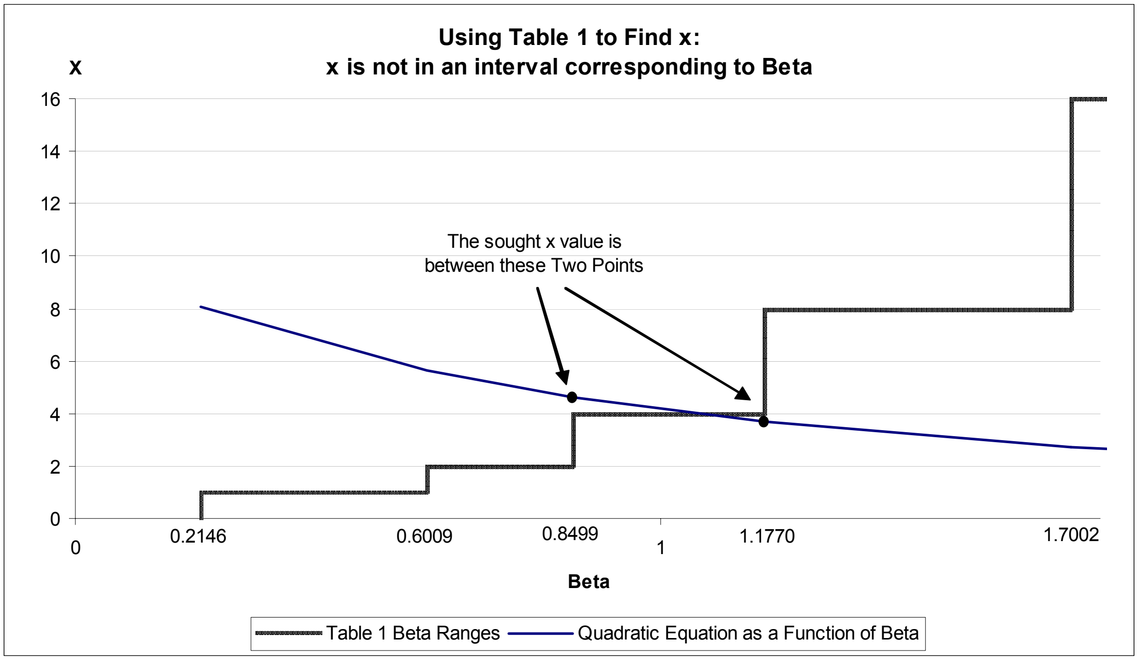


Figure 4: Using Table 1 to Find x: x is not in an interval corresponding to β_X .

6. The Multi-Family Algorithm

Algorithm P:

Step 1: Compute the tentative Item Multipliers

Define $\tilde{\mathbf{X}} = \left[\tilde{X}_{ij}\right]$, the set of tentative relaxed item multipliers.

For all (i, j) $\in$ $\mathbf{N}\times\mathbf{N}$, compute (41) below, with the approximate function solution procedure in §5.

$$\tilde{X}_{ij} = \frac{-\beta_X g_{ij} + \sqrt{\left(\beta_X g_{ij}\right)^2 + 4a_{ij}b_{ij}}}{2b_{ij}} \tag{41}$$

Step 2: Obtain the Family Multipliers and Finalize Item Multipliers

Define $\mathbf{X} = \left[X_{ij}\right]$, the set of finalized relaxed item multipliers.

Re-index $\tilde{\mathbf{X}}$ so that $\forall \tilde{X}_{i\bullet} \in \tilde{\mathbf{X}}, \tilde{X}_{i1} \le \tilde{X}_{i2} \le \cdots \le \tilde{X}_{iN}$. For notational convenience, let $\tilde{X}_{i,N+1} = \tilde{X}_{iN}$. Perform the following routine, $\forall$ i $\in$ **N :**

For h=1 to N loop {

Compute

$$\bar{a}_{ih} = A_i + \sum\nolimits_{j=1}^{h} a_{ij}, \tag{42}$$

$$\bar{b}_{ih} = \sum\nolimits_{j=1}^{h} b_{ij}, \text{ and} \tag{43}$$

$$\bar{g}_{ih} = \sum\nolimits_{j=1}^{h} g_{ij}. \tag{44}$$

Using the function approximation solution procedure in §5, compute(45).

$$Y_i = \frac{-\beta_X \bar{g}_{ij} + \sqrt{(\beta_X \bar{g}_{ij})^2 + 4\bar{a}_{ij}\bar{b}_{ij}}}{2\bar{b}_{ij}} \tag{45}$$

If $\tilde{X}_{ih} = Y_i \le \tilde{X}_{i,h+1}$, then

Set $X_{ij} = Y_i, \forall\, j = 1,\cdots,h.$

Set $X_{ij} = \tilde{X}_{ij}, \forall\, j = h+1,\cdots,N.$

Break.

Else $\left(\exists\, j > h \ni X_{ij} < Y_i\right.$, which violates contraint (25)$\left.\right)$,

Continue.

End if

Loop}.

Step 3: Round-off to Integer-Powers-of-Two

This step is motivated by Roundy's Roundoff Algorithm (1988).

Define $\mathbf{X}_{i\cdot} = \left(X_{ij}^{h} : (i,j) \in \{i\} \times \mathbf{N};\ h=1,\cdots,N\right)$, $\mathbf{Y}^{f} = \left(Y_{i}^{f} : \text{computed in iteration f}\right)$, and

$\mathbf{X}^{f} = \left[X_{ij}^{f} : \text{computed in iteration f}\right]$.

For all items $(i,j) \in \mathbf{N} \times \mathbf{N}$, find z_{ij} and p_{ij} so that

$$X_{ij} = z_{ij} \cdot 2^{p_{ij}},\ \ni z_{ij} \in [d, 2d), \tag{46}$$

For some $d \in \left(\frac{1}{2}, 1\right]$. Note the uniqueness of z_{ij} and p_{ij}.

Re-index $\mathbf{X}_{i\cdot}$, $\forall\, i \in \mathbf{N}$, so that $X_{ij}^{h} \le X_{ij}^{h+1}$, $h=1,\cdots,N$.

For f =1 to N^2 loop {

$$\forall\ h = 1,\ldots,N^2, \text{compute: } q_{ij}^{h} = \begin{cases} \max\left\{ p_{ij}^{h} - 1,\ 0 \right\}, & \text{if } h \le f \\ p_{ij}^{h}, & \text{otherwise.} \end{cases} \tag{47}$$

Compute:

$$k_{ij}^{h} = 2^{q_{ij}^{h}}. \tag{48}$$

Set $\mathbf{k}^{f} = \left[k_{ij}^{h} : h = 1,\cdots,N^2\right]$.

For each family i, compute:

$$K_{i}^{h} = \min\left\{k_{ij}^{h} : k_{ij}^{h} \in X_{i\cdot}\right\}. \tag{49}$$

Set $\mathbf{K}^{f} = \left(K_{i}^{h} : h = 1,\cdots,N^2\right)$.

Compute:

$$\bar{A}_i = \frac{A_i}{K_i^h} + \sum_{j \in \mathbf{N}} \frac{a_{ij}}{k_{ij}^{h}}, \tag{50}$$

$$\bar{b}_i = \sum_{j\in\mathbf{N}} b_{ij} \cdot k_{ij}^h, \text{ and} \tag{51}$$

$$\bar{g}_i = \sum_{j\in\mathbf{N}} g_{ij} \cdot \sqrt{k_{ij}^h}. \tag{52}$$

Next, compute:

$$\bar{A} = \sum_{i\in\mathbf{N}} \bar{A}_i, \tag{53}$$

$$\bar{b} = \sum_{i\in\mathbf{N}} \bar{b}_i, \text{ and} \tag{54}$$

$$\bar{g} = \sum_{i\in\mathbf{N}} \bar{g}_i. \tag{55}$$

Use $\bar{A}$, $\bar{b}$, and $\bar{g}$ and using the function approximation solution procedure in §5 to compute

$$\alpha^f = \frac{-\beta_X\bar{g} + \sqrt{(\beta_X\bar{g})^2 + 4\bar{A}\,\bar{b}}}{2\bar{b}}. \tag{56}$$

Compute the value of γ^f that binds G(X, Y) in constraint (24), i.e.

$$\gamma^f = \frac{\sum_{i\in\mathbf{N}} \frac{S_i}{K_i^h} + \sum_{i\in\mathbf{N}} \sum_{j\in\mathbf{N}} \frac{s_{ij}}{k_{ij}^h}}{(1-\rho)}. \tag{57}$$

Finally, select the basic period length $\beta^f = \max(\alpha^f, \gamma^f)$, and compute $C^f\left(\beta^f, \mathbf{K}^f, \mathbf{k}^f\right)$.

} loop.

The solution is $(T', \mathbf{K}', \mathbf{k}') = \operatorname{argmin}\left\{C^f\left(\beta^f, \mathbf{K}^f, \mathbf{k}^f\right) : f = 1, \cdots, N^2\right\}$.

The objective value is $C'(T', \mathbf{K}', \mathbf{k}') = \min\left\{C^f\left(\beta^f, \mathbf{K}^f, \mathbf{k}^f\right) : f = 1, \cdots, N^2\right\}$.

Step 3a: Tweak Integer-Powers-of-Two Upward

Let $C^{\text{incumbent}} = C'(T', \mathbf{K}', \mathbf{k}')$, $T^{\text{incumbent}} = T'$, and $\mathbf{p}^{\text{incumbent}} = \left[p_{ij} : 2^{p_{ij}} = k_{ij} \in \mathbf{k}'\right]$.

For f =1 to N^2 loop {

$$\forall\ h = 1,\ldots, N^2, \text{ compute: } q_{ij}^h = \begin{cases} p_{ij}^h + 1, & \text{if } h = f \\ p_{ij}^h, & \text{otherwise.} \end{cases} \tag{58}$$

Let $\mathbf{k}^f = \left[k_{ij}^h : k_{ij}^h = 2^{q_{ij}^h} \right]$.

Set $K_i^h = \max\left\{ k_{ij}^h : k_{ij}^h \in k_{i\cdot}^h, \forall\ k_{i\cdot}^h \in \mathbf{k}^f \right\}, \forall\ i \in \mathbf{N}$.

Let $\mathbf{K}^f = \left(K_i^h : i \in \mathbf{N} \right)$.

For each family i, compute:

$$\bar{A}_i = \frac{A_i}{K_i^h} + \sum_{j \in \mathbf{N}} \frac{a_{ij}}{k_{ij}^h}, \tag{59}$$

$$\bar{b}_i = \sum_{j \in \mathbf{N}} b_{ij} \cdot k_{ij}^h, \text{ and} \tag{60}$$

$$\bar{g}_i = \sum_{j \in \mathbf{N}} g_{ij} \cdot \sqrt{k_{ij}^h}. \tag{61}$$

Next, compute:

$$\bar{A} = \sum_{i \in \mathbf{N}} \bar{A}_i, \tag{62}$$

$$\bar{b} = \sum_{i \in \mathbf{N}} \bar{b}_i, \text{ and} \tag{63}$$

$$\bar{g} = \sum_{i \in \mathbf{N}} \bar{g}_i. \tag{64}$$

Use $\bar{A}$, $\bar{b}$, and $\bar{g}$ and using the function approximation solution procedure in §5 to compute

$$\alpha^f = \frac{-\beta_X \bar{g} + \sqrt{(\beta_X \bar{g})^2 + 4\bar{A}\bar{b}}}{2\bar{b}}. \tag{65}$$

Compute the value of Y^f that binds G(X, Y) in constraint (24), i.e.

$$\gamma^f = \frac{\sum_{i\in\mathbf{N}} \frac{S_i}{K_i^h} + \sum_{i\in\mathbf{N}}\sum_{j\in\mathbf{N}} \frac{s_{ij}}{k_{ij}^h}}{(1-\rho)}. \tag{66}$$

Finally, select the basic period length $\beta^f = \max(\alpha^f, \gamma^f)$, and compute $C^f\left(\beta^f, \mathbf{K}^f, \mathbf{k}^f\right)$.

If $C^f\left(\beta^f, \mathbf{K}^f, \mathbf{k}^f\right) < C^{incumbent}$, then

$C^{incumbent} \leftarrow C^f\left(\beta^f, \mathbf{K}^f, \mathbf{k}^f\right)$,

$T^{incumbent} \leftarrow \beta^f$,

$\mathbf{p}^{incumbent} \leftarrow \left[q_{ij}^h : h = 1, \cdots N^2\right]$.

End if

} Loop.

$$C^* \leftarrow C^{incumbent}, \tag{67}$$

$$T^* \leftarrow T^{incumbent}, \tag{68}$$

$$\mathbf{p}^* \leftarrow \mathbf{p}^{incumbent}. \tag{69}$$

Step 3b: Tweak Integer-Powers-of-Two Downward

Repeat step 3a, replacing equation (58) with (70) below.

$$\forall\ h = 1,\ldots, N^2, \text{compute: } q_{ij}^h = \begin{cases} p_{ij}^h - 1, & \text{if } h = f \\ \\ p_{ij}^h, & \text{otherwise.} \end{cases} \tag{70}$$

Step 4: Solution $\left(T^*, \mathbf{K}^*, \mathbf{k}^*\right)$

With T* given in (68) and

$$\mathbf{k}^* = \left[k_{ij}^* = 2^{p_{ij}} : p_{ij} \in \mathbf{p}^*\right], \tag{71}$$

$$\mathbf{K}^* = \left(K_i^* : K_i^* = \min\left\{k_{ij}^* : k_{ij}^* \in \mathbf{k}^*\right\}, \forall\ i \in \mathbf{N}\right). \tag{72}$$

The objective value is given by (67) above.

The solution is then rendered in AF for ease of implementation. For each $i \in \mathbf{N}$, set

$$K_i = \frac{K_i^*}{\min\left\{K_i^* : i \in \mathbf{N}\right\}}. \quad \text{Then, } T = T^* \min\left\{K_i^* : i \in \mathbf{N}\right\}.$$

7. Example

We illustrate our multi-family scheduling solution procedure with an example. In this example, a facility produces five families, each with five items. The time unit was arbitrarily chosen to be one week. The data were generated from uniformly distributed parameters as follows:

Parameter	Distribution
Family setup time	U(0.015, 0.025)
Family setup cost	U(100, 500)
Item setup time	U(0.0012, 0.018)
Item setup cost	U(50, 150)
Item holding cost	U(0.01, 1.25)
Item demand mean	U(10, 500)
Item demand standard deviation (% of demand mean)	U(0.5, 0.85)
Item production rate	U(10,000, 15,000)
Item service level	U(0.85, 0.9999)

The data are:

Family setup times: $\mathbf{S} = \begin{pmatrix} 0.0165 & 0.0183 & 0.0209 & 0.0244 & 0.0240 \end{pmatrix}$

Family setup costs: $\mathbf{A} = \begin{pmatrix} \$396 & \$269 & \$352 & \$125 & \$435 \end{pmatrix}$

Item setup times:

$$\mathbf{s} = \begin{pmatrix} 0.0089 & 0.0159 & 0.0039 & 0.0124 & 0.0179 \\ 0.0056 & 0.0079 & 0.0066 & 0.0175 & 0.0069 \\ 0.0109 & 0.0090 & 0.0131 & 0.0149 & 0.0087 \\ 0.0127 & 0.0031 & 0.0179 & 0.0078 & 0.0054 \\ 0.0043 & 0.0038 & 0.0138 & 0.0117 & 0.0017 \end{pmatrix}$$

Item setup costs: $\mathbf{a} = \begin{pmatrix} \$93 & \$84 & \$74 & \$125 & \$150 \\ \$96 & \$105 & \$139 & \$139 & \$88 \\ \$81 & \$131 & \$143 & \$112 & \$75 \\ \$89 & \$105 & \$57 & \$61 & \$131 \\ \$107 & \$65 & \$102 & \$72 & \$124 \end{pmatrix}$

Item holding cost: $\mathbf{h} = \begin{pmatrix} \$0.96 & \$0.09 & \$0.21 & \$1.20 & \$0.57 \\ \$0.70 & \$0.47 & \$0.70 & \$0.11 & \$0.62 \\ \$1.16 & \$0.73 & \$1.02 & \$1.18 & \$0.63 \\ \$1.10 & \$0.42 & \$0.86 & \$0.45 & \$0.08 \\ \$0.56 & \$0.39 & \$0.50 & \$0.13 & \$0.06 \end{pmatrix}$

Item demand mean: $\mathbf{d} = \begin{pmatrix} 96 & 17 & 448 & 379 & 328 \\ 73 & 124 & 78 & 474 & 56 \\ 309 & 245 & 458 & 381 & 361 \\ 179 & 419 & 110 & 78 & 337 \\ 250 & 290 & 494 & 43 & 484 \end{pmatrix}$

Item demand standard deviation:

$$\boldsymbol{\sigma} = \begin{pmatrix} 65.79 & 9.23 & 379.77 & 265.45 & 200.31 \\ 36.97 & 79.67 & 63.48 & 353.84 & 30.88 \\ 218.68 & 173.71 & 342.47 & 278.21 & 245.16 \\ 135.13 & 315.63 & 56.74 & 57.88 & 210.56 \\ 193.90 & 188.53 & 268.93 & 33.43 & 322.63 \end{pmatrix}$$

Item production rate:

$$\mathbf{p} = \begin{pmatrix} 12{,}549 & 13{,}008 & 11{,}089 & 14{,}356 & 14{,}876 \\ 13{,}619 & 13{,}774 & 14{,}794 & 11{,}220 & 11{,}739 \\ 11{,}853 & 12{,}614 & 14{,}133 & 13{,}875 & 14{,}111 \\ 12{,}843 & 10{,}267 & 12{,}774 & 12{,}358 & 12{,}014 \\ 12{,}388 & 12{,}691 & 11{,}722 & 12{,}029 & 10{,}408 \end{pmatrix}$$

Item service level:

$$\mathbf{SL} = \begin{pmatrix} 0.8520 & 0.9141 & 0.9553 & 0.9949 & 0.9704 \\ 0.9806 & 0.9741 & 0.8650 & 0.9912 & 0.8833 \\ 0.9446 & 0.8844 & 0.8517 & 0.9935 & 0.9225 \\ 0.8816 & 0.9014 & 0.9491 & 0.9465 & 0.8909 \\ 0.9401 & 0.9705 & 0.8960 & 0.8789 & 0.9476 \end{pmatrix}$$

Computation results are shown using two procedures. (1) We first use the multi-family algorithm as given in §6 above. (2) Next, we apply the multi-family algorithm to the corresponding deterministic problem, with demand means used as demand rates. Once the solution is obtained, we add safety stocks and compute their costs. The reason for using (2) for comparison is twofold. First, a deterministic algorithm for our problem does not exist in the literature. Second, we illustrate the importance of including the safety stock costs explicitly in the problem formulation by comparing the resultant average total cost to that resulting from the common approach of ignoring safety stock costs in the formulation while maintaining safety stock levels in practice.

Algorithm	(1)*	(2)**
Total average cost	TC=$9,834.72	$10,825.93
% Difference		10.08%
Total average Family setup cost	$1,597.13	$1,152.29.42
Total average Item setup cost	$2,296.86	$1,611.15
Total average working stock holding cost	$1,795.05	$2,817.35
Total average safety stock holding cost	$4,148.87	$5,200.67
Basic period length	T*=0.511	1.050
Family multipliers	$\mathbf{k}^*=(2,\ 2,\ 2,\ 1,\ 2)$	$(1,\ 1,\ 1,\ 1,\ 2)$
Item multipliers	$\mathbf{K}^*=\begin{bmatrix} 1 & 8 & 1 & 1 & 1 \\ 1 & 1 & 2 & 4 & 2 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 2 \\ 8 & 1 & 1 & 8 & 2 \end{bmatrix}$	$\begin{bmatrix} 1 & 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 2 & 2 \\ 4 & 1 & 1 & 2 & 1 \end{bmatrix}$
*(1) Multi-family algorithm **(2) Deterministic multi-family with safety stocks at end		

Using the multi-family algorithm, which considers safety stock costs explicitly, resulted in cost savings of 10.08% over algorithm (2). This represents annual savings of $51,542.92 over algorithm (2). The explicit consideration of safety stock costs in the MFELSP—SS was shown to be cost-beneficial. The average costs savings, over 200 randomly generated problems with the distribution of parameters given above, and the average savings were 12.39% over algorithm (2). This represents annual dollar savings of $ 111,982.84.

8. Summary

In this paper, we have extended the Multiple Family Economic Lot Sizing Problem to include the consideration of safety stocks explicitly in the optimization. We exploited a number of properties of the problem to develop an efficient solution procedure. The form of our solution is a basic period cyclic schedule. In addition, we restrict the family and item multipliers in the solution to be integer powers of 2. We solved a representative set of sample problems using our procedure, and compared the results to a solution procedure that ignored the safety stock costs. Our results exhibited total costs typically 10 to 12% lower than the alternative procedure. The alternative procedure is a relevant benchmark, since many companies are likely establishing economic production quantities for such product families using procedures that ignore the cost of carrying the safety inventories.

There are a number of directions for future research emanating from this first piece. We need to develop procedures to take the solution for our problem $(\mathrm{T}, \mathbf{K}, \mathbf{k})$ and actually construct a feasible schedule of assignments of families and products to each of the basic periods in a cycle to ensure that we are observing facility capacity constraints. Another problem is the dynamic problem of how to adjust production quantities during the production of products in the family when the state of the inventories to start the production run is not what was expected. This will typically be the case due to the uncertainties that exist in the problem.

Another avenue to explore is alternative service objectives than the Type I service level studied in this paper. One such alternative is fill rate, which is a criterion often used in distribution environments. Still other relaxations could involve the consideration of time varying patterns of demand for products, including trend and seasonal demand patterns.

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