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On Convergent Sequences of Linear Programs

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Abstract

This paper provides results pertaining to solving a sequence of linear programming (LP) problems. Specifically, given the data for a sequence of approximating LPs that converge to data for a limiting LP, several sufficient conditions are presented under which (a) the sequence of optimal objective function values for the approximating problems converges to the optimal objective function value of the limiting problem (even though the optimal solutions might not converge) and (b) any convergent subsequence of optimal solutions to the approximating problems converges to an optimal solution of the limiting problem.

1 Introduction and Notation

This paper is motivated by a variety of applications in which it is necessary to solve a sequence of linear programming (LP) problems. In such applications, the optimal solution to the current LP is used to create data for the next LP, as summarized in the following general model.

A General Sequential Linear Programming Model

Step 0: Let $\mathbf{c}^1$ be a given n-vector, $\mathbf{b}^1$ be a given m-vector, and A^1 be a given $(m \times n)$ matrix for which the following LP^1 has an optimal solution and set $k = 1$:

$$\begin{array}{llll} \max & \mathbf{c}^1\mathbf{x} & & \\ \text{subject to} & A^1\mathbf{x} & \leq & \mathbf{b}^1 \end{array} \qquad (\text{LP}^1)$$

Step 1: Let $\mathbf{x}^k$ be an optimal solution to LP^k.

Step 2: Use $\mathbf{x}^k$ and appropriate update formulas to compute $\mathbf{c}^{k+1}$, $\mathbf{b}^{k+1}$, and a matrix A^{k+1} for which LP^{k+1} has an optimal solution. Set $k = k + 1$ and go to Step 1.

Solow et al. (2001) use such a model to study the emergence of *functional specialization*—the property that, over time, individuals in a community tend to spend virtually all of their time performing one, or at most a few, tasks. Another such application arises in Solow and Sengupta (1985) and Solow and Papparizos (1989) to solve the Linear Complementarity Problem, however, in those cases, the sequence of LPs is finite.

A sequence of such problems also arises when the data to a limiting LP are not known directly, but rather, are approximated through a sequence of LPs. In Derman (1970) it is shown that a stochastic dynamic programming problem with finite states and finite actions can be formulated and solved as a linear programming problem. In such a linear program the A^k matrix is a function of the estimate of the transition probabilities between states at iteration k. As solutions to the linear program are implemented, additional information regarding the transition probabilities is gained. This additional information is then used to update the estimates for the transition probabilities between states. The resulting estimates are then used in a new formulation with constraints corresponding to A^{k+1}. Solutions to the new linear program are then implemented and the process repeated, resulting in a sequence of linear programming problems.

A final example of the need to solve a sequence of LPs arises in sampling-based methods to solve two-stage stochastic programming problems [see, for example, Birge (1997)]. In this application, the constraints of the master problems are sequentially refined by using the optimal solutions to a collection of second-stage LPs.

To study general sequences of LPs, for each $k = 1, 2, \ldots$, let

$\mathbf{c}^k$ = an n-vector representing the objective function coefficients of LP^k,
$\mathbf{b}^k$ = an m-vector representing the right-hand-side values of LP^k, and
A^k = the $(m \times n)$ matrix of constraint coefficients for LP^k.

Then the associated LP^k is:

$$\begin{array}{lrcl} \max & \mathbf{c}^k\mathbf{x} & & \\ \text{subject to} & A^k\mathbf{x} & \leq & \mathbf{b}^k \end{array} \qquad (\mathrm{LP}^k)$$

It is assumed that $(\mathbf{c}^k, \mathbf{b}^k, A^k) \to (\mathbf{c}^\infty, \mathbf{b}^\infty, A^\infty)$, where $\mathbf{c}^\infty$, $\mathbf{b}^\infty$, and A^∞ are the data for the limiting LP$^\infty$ and (A^k) converges to A^∞ element by element, that is, for all i and j, $(A^k_{ij}) \to A^\infty_{ij}$ as $k \to \infty$.

1.1 Examples and Counter Examples

When looking at geometric examples of such convergent sequences of LPs, two types of results appear reasonable: (1) the optimal objective function values of LP^k converge to the optimal objective function value of LP$^\infty$ and (2) any convergent subsequence of optimal solutions to LP^k converges to an optimal solution of LP$^\infty$. The following examples show that these intuitive results may or may not hold.

Example 1: A Sequence of Optimal LPs that Converge to an Optimal LP in Which the Optimal Objective Function Values and Solutions Converge.

$$
\begin{array}{ll}
\underline{\text{LP}^k \quad \text{(Optimal)}} & \\
\max & x_1 + x_2 \\
\text{s.t.} & \frac{1}{k}x_1 \le 1 \\
& \frac{1}{k}x_2 \le 1 \\
& x_1 \le 1 \\
& x_2 \le 1
\end{array}
\qquad
\begin{array}{ll}
\underline{\text{LP}^\infty \quad \text{(Optimal)}} & \\
\max & x_1 + x_2 \\
\text{s.t.} & 0x_1 \le 1 \\
& 0x_2 \le 1 \\
& x_1 \le 1 \\
& x_2 \le 1
\end{array}
$$

$$(x_1^k, x_2^k) = (1,1), \quad x_1^k + x_2^k = 2 \qquad (x_1^\infty, x_2^\infty) = (1,1), \quad x_1^\infty + x_2^\infty = 2$$

However, the next two examples show that this need not always happen.

Example 2: A Sequence of Optimal LPs that Converge to an Optimal LP in Which the Optimal Objective Function Values Do Not Converge.
Neither the optimal solutions nor the optimal objective function values of the following sequence of LPs converges to that of the limiting LP:

$$
\begin{array}{ll}
\underline{\text{LP}^k \quad \text{(Optimal)}} & \\
\max & x_1 + x_2 \\
\text{s.t.} & \frac{1}{k}x_1 \le 0 \\
& \frac{1}{k}x_2 \le 0 \\
& -1 \le x_1 \le 1 \\
& -1 \le x_2 \le 1
\end{array}
\qquad
\begin{array}{ll}
\underline{\text{LP}^\infty \quad \text{(Optimal)}} & \\
\max & x_1 + x_2 \\
\text{s.t.} & 0x_1 \le 0 \\
& 0x_2 \le 0 \\
& -1 \le x_1 \le 1 \\
& -1 \le x_2 \le 1
\end{array}
$$

$$(x_1^k, x_2^k) = (0,0), \quad x_1^k + x_2^k = 0 \qquad (x_1^\infty, x_2^\infty) = (1,1), \quad x_1^\infty + x_2^\infty = 2$$

Example 3: Another Sequence of Optimal LPs that Converge to an Optimal LP in Which the Optimal Objective Function Values Do Not Converge.

$$
\begin{array}{ll}
\underline{\text{LP}^k \quad \text{(Optimal)}} & \\
\max & x_1 \\
\text{s.t.} & x_1 - \frac{1}{k}x_2 \le 0 \\
& x_1 \le 1
\end{array}
\qquad
\begin{array}{ll}
\underline{\text{LP}^\infty \quad \text{(Optimal)}} & \\
\max & x_1 \\
\text{s.t.} & x_1 - 0x_2 \le 0 \\
& x_1 \le 1
\end{array}
$$

$$(x_1^k, x_2^k) = (1,k), \quad x_1^k = 1 \qquad (x_1^\infty, x_2^\infty) = (0,1), \quad x_1^\infty = 0$$

In general, LP^∞ can be infeasible, optimal, or unbounded. When LP^∞ is infeasible, it is possible to have a sequence of infeasible LPs that converge to LP^∞, as shown in the next example.

Example 4: A Sequence of Infeasible LPs that Converge to an Infeasible LP.

$$
\begin{array}{ll}
\underline{\text{LP}^k \quad \text{(Infeasible)}} & \\
\max & x_1 \\
\text{s.t.} & 1 \le x_1 \le -\frac{1}{k}
\end{array}
\qquad
\begin{array}{ll}
\underline{\text{LP}^\infty \quad \text{(Infeasible)}} & \\
\max & x_1 \\
\text{s.t.} & 1 \le x_1 \le 0
\end{array}
$$

The next example shows that is also possible to have a sequence of feasible LPs that converge to an infeasible LP.

Example 5: A Sequence of Feasible LPs that Converge to an Infeasible LP.

$$\begin{array}{llll} \underline{\text{LP}^k \quad \text{(Feasible)}} & & & \\ \max & x_1 & & \\ \text{s.t.} & \frac{1}{k}x_1 - x_2 & \geq & 1 \\ & x_1 \; , \; x_2 & \geq & 0 \end{array} \qquad \begin{array}{llll} \underline{\text{LP}^\infty \quad \text{(Infeasible)}} & & & \\ \max & x_1 & & \\ \text{s.t.} & 0x_1 - x_2 & \geq & 1 \\ & x_1 \; , \; x_2 & \geq & 0 \end{array}$$

Turning to the case when LP^∞ is unbounded, the following three examples show that the desired convergence results may or may not hold.

Example 6: A Sequence of Infeasible LPs that Converge to an Unbounded LP.

$$\begin{array}{llll} \underline{\text{LP}^k \quad \text{(Infeasible)}} & & & \\ \max & x_2 & & \\ \text{s.t.} & x_1 & \leq & -\frac{1}{k} \\ & x_1, \; x_2 & \geq & 0 \end{array} \qquad \begin{array}{llll} \underline{\text{LP}^\infty \quad \text{(Unbounded)}} & & & \\ \max & x_2 & & \\ \text{s.t.} & x_1 & \leq & 0 \\ & x_1, \; x_2 & \geq & 0 \end{array}$$

Example 7: A Sequence of Optimal LPs that Converge to an Unbounded LP.

$$\begin{array}{llll} \underline{\text{LP}^k \quad \text{(Optimal)}} & & & \\ \max & x_1 & & \\ \text{s.t.} & \frac{1}{k}x_1 & \leq & 1 \end{array} \qquad \begin{array}{llll} \underline{\text{LP}^\infty \quad \text{(Unbounded)}} & & & \\ \max & x_1 & & \\ \text{s.t.} & 0x_1 & \leq & 1 \end{array}$$

$$x_1^k = k$$

Example 8: A Sequence of Unbounded LPs that Converge to an Unbounded LP.

$$\begin{array}{llll} \underline{\text{LP}^k \quad \text{(Unbounded)}} & & & \\ \max & x_2 & & \\ \text{s.t.} & x_1 & \leq & \frac{1}{k} \\ & x_1, \; x_2 & \geq & 0 \end{array} \qquad \begin{array}{llll} \underline{\text{LP}^\infty \quad \text{(Unbounded)}} & & & \\ \max & x_2 & & \\ \text{s.t.} & x_1 & \leq & 0 \\ & x_1, \; x_2 & \geq & 0 \end{array}$$

1.2 Notation

The examples in Section 1.1 indicate that conditions are needed to establish the desired convergence results. A number of such conditions are presented in Sections 2 and 3 along with answers to the following questions:

1. Under what conditions does the sequence of optimal objective function values converge to the optimal objective function value of the limiting LP (even though the sequence of optimal solutions might not converge)?

2. Under what conditions does any convergent subsequence of optimal solutions to the approximating LPs converge to an optimal solution of the limiting LP?

To simplify the analysis, it is now shown that it suffices to study a sequence of LPs in which the objective function coefficients and right-hand-side values are the same from one problem to the next and only the A matrix changes. To that end, introducing two new variables y and z, note that LP^k is equivalent to the following LP, in which only the coefficient matrix changes:

$$\begin{array}{lrcrcrcl} \max & & & & & z & & \\ \text{s.t.} & \mathbf{c}^k\mathbf{x} & - & & & z & = & 0 \\ & A^k\mathbf{x} & - & \mathbf{b}^k y & & & \leq & \mathbf{0} \\ & & & y & & & = & 1 \end{array}$$

As a result of the foregoing equivalence, consider now a sequence of LPs in which only the coefficient matrix A^k changes from one problem to the next.

To achieve the desired convergence results, the following LPs are used. For given vectors $\mathbf{c} \in R^n$, $\mathbf{b} \in R^m$, and $\mathbf{e} = (1, \ldots, 1) \in R^m$, define the following primal and dual LPs, for every $(m \times n)$ matrix A and real number $\epsilon \geq 0$:

$$\begin{array}{llcl} \underline{\text{Primal LP}(A,\, \epsilon)} & & & \\ \max & \mathbf{cx} & & \\ \text{s.t.} & A\mathbf{x} & \leq & \mathbf{b} + \epsilon\mathbf{e} \end{array} \qquad \begin{array}{llcl} \underline{\text{Dual DLP}(A,\, \epsilon)} & & & \\ \min & \mathbf{ub} + \epsilon\mathbf{ue} & & \\ \text{s.t.} & A^T\mathbf{u} & = & \mathbf{c} \\ & \mathbf{u} & \geq & \mathbf{0} \end{array}$$

For clarity of notation, here and throughout the paper, transpose notation for vectors is omitted and understood from the context in which the vectors are used. However, transpose notation is used for matrices, where appropriate.

Note that $\text{LP}(A^k,\, 0)$ is the k^{th} primal LP in the original sequence and $\text{LP}(A^\infty,\, 0)$ is the limiting primal LP. When $\epsilon > 0$, $\text{LP}(A,\, \epsilon)$ is the **ϵ-outer LP for LP(A)**, as depicted in Figure 1. Whenever $\text{LP}(A,\, \epsilon)$ is optimal, let $X(A, \epsilon)$ be the set of optimal solutions for $\text{LP}(A,\, \epsilon)$ and let $\mathbf{x}(A,\, \epsilon)$ be any element in $X(A, \epsilon)$ and $\mathbf{u}(A,\, \epsilon) \in R^m$ be any optimal solution for $\text{DLP}(A,\, \epsilon)$. Also, the following observation is used frequently.

Observation 1.1 *If A is a matrix for which LP(A, 0) is optimal, then $\forall \epsilon \geq 0$, the feasible region of DLP(A, ϵ) is the same nonempty set.*

2 Preliminary Convergence Results

As a result of the examples in Section 1.1, it is now assumed that for all k sufficiently large, $\text{LP}(A^k,\, 0)$ is optimal, that is, there is an integer $\bar{k}$ such that for all $k > \bar{k}$, $\text{LP}(A^k,\, 0)$ is optimal. To simplify the notation in the subsequent analysis, by dropping the first $\bar{k}$ problems and then renumbering, from here on it is assumed that:

Assumption 1: For each $k = 1, 2, \ldots$, $\mathbf{x}(A^k,\, 0)$ is an optimal solution for $\text{LP}(A^k,\, 0)$ and $\mathbf{x}(A^\infty,\, 0)$ is an optimal solution for $\text{LP}(A^\infty,\, 0)$.

Note that the sequence $(\mathbf{x}(A^k,\, 0))$ need not converge to $\mathbf{x}(A^\infty,\, 0)$. Several sufficient conditions are presented in this section to ensure that the optimal objective function values of

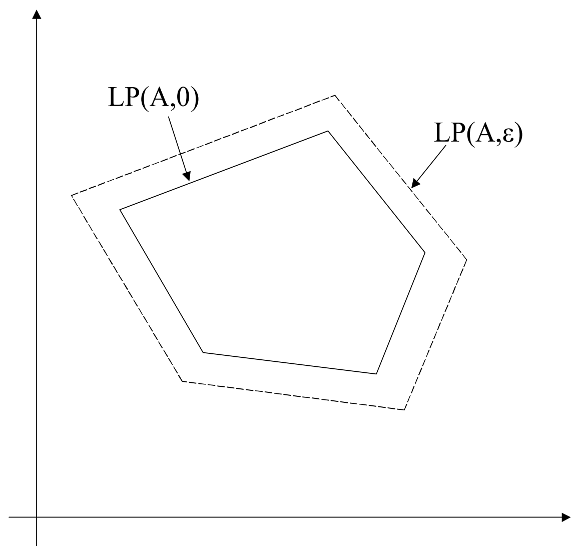

Figure 1: A Linear Program and its ϵ-Outer Linear Program.

LP(A^k, 0) converge to that of LP(A^∞, 0), that is, $(\mathbf{cx}(A^k, 0)) \to \mathbf{cx}(A^\infty, 0)$, even though the optimal solutions might not converge. The approach to doing so is to show that for every real number $\delta > 0$ and k sufficiently large, the following two inequalities hold:

$$\mathbf{cx}(A^k, 0) < \mathbf{cx}(A^\infty, 0) + \delta \tag{1}$$

and

$$\mathbf{cx}(A^k, 0) > \mathbf{cx}(A^\infty, 0) - \delta. \tag{2}$$

Besides Assumption 1, additional assumptions are needed to establish the inequalities in (1) and (2) and hence rule out the counter examples in Section 1.1. Such sufficient conditions are presented in Sections 2.1 and 2.2.

2.1 Sufficient Conditions for the Inequality $\mathbf{cx}(A^k, 0) < \mathbf{cx}(A^\infty, 0) + \delta$.

The approach used to establish the inequality in (1) requires that for every $\epsilon > 0$, the optimal solution to LP(A^k, 0) eventually be feasible for the ϵ-outer LP of LP(A^∞, 0). A sufficient condition for this to happen is that there be an integer $\bar{k}$ such that for all $k > \bar{k}$, the sequence $(\mathbf{x}(A^k, 0))$ lies inside a compact set. To simplify the notation in the subsequent analysis, by dropping the first $\bar{k}$ problems and then renumbering, it can be assumed that:

Assumption 2: There is a compact set $C \subset R^n$ such that the specific chosen sequence $(\mathbf{x}(A^k, 0)) \subseteq C$.

Indeed, if Assumption 2 does not hold, then the optimal objective function values for the approximating LPs need not converge to the optimal objective function value of the limiting LP, as shown in Example 3 in Section 1.1.

With Assumptions 1 and 2, it is now possible to establish that for every $\epsilon > 0$ and k sufficiently large, the optimal solution to LP(A^k, 0) is feasible for the ϵ-outer LP of LP(A^∞, 0).

Lemma 2.1 *Suppose that* $(A^k) \to A^\infty$ *and for every* $k = 1, 2, \ldots$, $\mathbf{x}(A^k, 0)$ *is optimal for LP*(A^k, 0). *If there is a compact set* $C \subset R^n$ *such that* $(\mathbf{x}(A^k, 0)) \subseteq C$, *then* $\forall \epsilon > 0$, *there is an integer* j_1 *such that* $\forall k > j_1$, $\mathbf{x}(A^k, 0)$ *is feasible for LP*(A^∞, ϵ).

Proof. Let $\epsilon > 0$ and note that if $A^\infty \mathbf{x}(A^k, 0) - A^k \mathbf{x}(A^k, 0) \leq \epsilon \mathbf{e}$, then $\mathbf{x}(A^k, 0)$ is feasible for LP(A^∞, ϵ) because

$$\begin{aligned} A^\infty \mathbf{x}(A^k, 0) &\leq A^k \mathbf{x}(A^k, 0) + \epsilon \mathbf{e} && \text{(by assumption)} \\ &\leq \mathbf{b} + \epsilon \mathbf{e} && [\mathbf{x}(A^k, 0) \text{ is feasible for LP}(A^k, 0)] \end{aligned}$$

It is now shown that there exists j_1 such that for all $k > j_1$, $A^\infty \mathbf{x}(A^k, 0) - A^k \mathbf{x}(A^k, 0) \leq \epsilon \mathbf{e}$, which will complete the proof. From here on, let $A^k_{i.}$ denote row i of matrix A^k and $A^\infty_{i.}$ denote row i of matrix A^∞.

Because $(\mathbf{x}(A^k, 0)) \subseteq C$ and C is compact, there is a real number $M > 0$ such that $\|\mathbf{x}(A^k, 0)\| < M$ for all k. Also, because $(A^k) \to A^\infty$, for each row $i = 1, \ldots, m$, $||A^\infty_{i.} - A^k_{i.}|| \to 0$. It follows that for each $i = 1, \ldots, m$, there is an integer k_i such that $||A^\infty_{i.} - A^k_{i.}|| \leq \epsilon/M$ for all $k > k_i$. Let $j_1 = \max_i k_i$. Then for $k > j_1$ and for each row $i = 1, \ldots, m$, $||A^\infty_{i.} - A^k_{i.}|| \leq \epsilon/M$ and so

$$\begin{aligned} A^\infty_{i.} \mathbf{x}(A^k, 0) - A^k_{i.} \mathbf{x}(A^k, 0) &= (A^\infty_{i.} - A^k_{i.}) \mathbf{x}(A^k, 0) \\ &\leq ||A^\infty_{i.} - A^k_{i.}|| \; ||\mathbf{x}(A^k, 0)|| \\ &\leq ||A^\infty_{i.} - A^k_{i.}|| M && \text{(by definition of } C \text{ and } M) \\ &\leq \epsilon && \text{(by the choice of } j_1) \end{aligned}$$

Now i is arbitrary in the foregoing inequality, so $A^\infty \mathbf{x}(A^k, 0) - A^k \mathbf{x}(A^k, 0) \leq \epsilon \mathbf{e}$ for all $k > j_1$. As has already been shown, this implies that $A^\infty \mathbf{x}(A^k, 0) \leq \mathbf{b} + \epsilon \mathbf{e}$ and so $\mathbf{x}(A^k, 0)$ is feasible for LP(A^∞, ϵ) and the proof is complete. □

It can be challenging to verify Assumption 2 in a specific application. However, in the following theorem, it is shown that if the feasible region of the limiting LP is compact, then Assumption 2 holds.

Theorem 2.1 *If* $(A^k) \to A^\infty$ *and the feasible region of LP*(A^∞, 0) *is compact, and*

Assumption 1: *For every* $k = 1, 2, \ldots$, $\mathbf{x}(A^k, 0)$ *is optimal for LP*(A^k, 0) *and* $\mathbf{x}(A^\infty, 0)$ *is optimal for LP*(A^∞, 0),

then there is a compact set $C \subset R^n$ *such that* $(\mathbf{x}(A^k, 0)) \subseteq C$.

Proof. Suppose, to the contrary, that the conclusion is not true. Then there is a subsequence K such that $\|\mathbf{x}(A^k, 0)\| \to \infty$ as $k \in K$. A contradiction is reached by showing that there is a direction $\mathbf{d} \in R^n$ with $\mathbf{d} \neq \mathbf{0}$ such that $A^\infty \mathbf{d} \leq \mathbf{0}$ and hence the feasible region of LP(A^∞, 0) is not compact. To that end, consider the sequence

$$\mathbf{d}^k = \frac{\mathbf{x}(A^k, 0)}{\|\mathbf{x}(A^k, 0)\|}, \quad \text{for } k \in K.$$

Because $\|\mathbf{d}^k\| = 1$ for all $k \in K$, there is a subsequence $K' \subseteq K$ and a vector $\mathbf{d} \in R^n$ with $\|\mathbf{d}\| = 1$ such that $(\mathbf{d}^k) \to \mathbf{d}$ as $k \in K'$. Now each $\mathbf{x}(A^k, 0)$ is feasible for LP(A^k, 0), so,

$$A^k \mathbf{d}^k = \frac{A^k \mathbf{x}(A^k, 0)}{\|\mathbf{x}(A^k, 0)\|} \leq \frac{\mathbf{b}}{\|\mathbf{x}(A^k, 0)\|}, \quad \text{for all } k \in K'. \tag{3}$$

Taking the limit as $k \in K'$ in (3) and using the facts that $(A^k) \to A^\infty$, $(\mathbf{d}^k) \to \mathbf{d}$, and $\|\mathbf{x}(A^k, 0)\| \to \infty$ as $k \in K'$, it follows that $A^\infty \mathbf{d} \leq \mathbf{0}$. Thus, it has been shown that there is a direction $\mathbf{d} \in R^n$ with $\mathbf{d} \neq \mathbf{0}$ such that $A^\infty \mathbf{d} \leq \mathbf{0}$ and hence the feasible region of LP(A^∞, 0) is not compact. This contradiction completes the proof. □

The proof of Theorem 2.1 applies to any sequence of feasible points for LP(A^k, 0) and therefore establishes that each LP(A^k, 0) is compact under the hypothesis of the theorem. With regard to verifying that hypothesis in an application, an explicit representation for the limiting LP is generally not available. However, in some cases, there is enough partial information to ensure that the feasible region is compact. For example, in the application in Solow et al. (2001), the limiting LP contains, among others, the constraints that the variables lie inside the unit simplex and hence the feasible region is compact.

As shown in the next theorem, for Assumption 2 to hold, it suffices for the set of optimal solutions of the limiting LP to be compact, provided that there is a lower bound on the optimal objective function values of the approximating LPs.

Theorem 2.2 *If $(A^k) \to A^\infty$, Assumption 1 holds, the set of optimal solutions to LP(A^∞, 0) is compact, and there is a real number z such that $\mathbf{cx}(A^k, 0) \geq z$ for all $k = 1, 2, \ldots$, then there is a compact set $C \subset R^n$ such that $(\mathbf{x}(A^k, 0)) \subseteq C$.*

Proof. Suppose that the conclusion of the theorem is not true. Then, as in the proof of Theorem 2.1, there is a subsequence K such that $\|\mathbf{x}(A^k, 0)\| \to \infty$ as $k \in K$ as well as a vector $\mathbf{d} \in R^n$ and a sequence of directions

$$\mathbf{d}^k = \frac{\mathbf{x}(A^k, 0)}{\|\mathbf{x}(A^k, 0)\|}, \quad \text{for } k \in K,$$

such that

$$(\mathbf{d}^k) \to \mathbf{d} \quad \text{as } k \in K, \tag{4}$$

$$\|\mathbf{d}^k\| = 1, \quad \forall k \in K, \tag{5}$$

$$\mathbf{d} \neq \mathbf{0}, \tag{6}$$

$$A^\infty \mathbf{d} \leq \mathbf{0}, \tag{7}$$

Now $\mathbf{d} \neq \mathbf{0}$ from (6) and $A^\infty \mathbf{d} \leq \mathbf{0}$ from (7), so $\mathbf{d}$ is a direction of recession for the feasible region of LP(A^∞, 0). It is now shown that $\mathbf{d}$ also satisfies $\mathbf{cd} \geq 0$ because, for each $k \in K$,

$$\begin{array}{lll} \mathbf{cx}(A^k,\,0) & \geq & z \qquad\qquad\qquad \text{(by hypothesis)} \\ \mathbf{cd}^k & \geq & z/\|\mathbf{x}(A^k,\,0)\| \quad \text{(divide by } \|\mathbf{x}(A^k,\,0)\| > 0). \end{array}$$

The fact that $\mathbf{cd} \geq 0$ follows by taking the limit as $k \in K$ on both sides of the last inequality.

It has now been shown that $\mathbf{d}$ is a nonzero direction in which it is possible to move infinitely far from $\mathbf{x}(A^\infty, 0)$, stay feasible for LP(A^∞, 0) (because $A^\infty \mathbf{d} \leq \mathbf{0}$), and not decrease the objective function (because $\mathbf{cd} \geq 0$). In other words, the set of optimal solutions to LP(A^∞, 0) is not compact. This contradiction completes the proof. □

Assumptions 1 and 2 are now used to establish the first inequality of the convergence result of the optimal objective function values.

Theorem 2.3 *If the sequence* $(A^k) \to A^\infty$ *and*

Assumption 1: *For every* $k = 1, 2, \ldots$, $\mathbf{x}(A^k,\,0)$ *is optimal for LP(*A^k, 0*) and* $\mathbf{x}(A^\infty,\,0)$ *is optimal for LP(*A^∞, 0*) and*

Assumption 2: *There is a compact set* $C \subset R^n$ *such that* $(\mathbf{x}(A^k,\,0)) \subseteq C$,

then $\forall \delta > 0$, *there is an integer* j_1 *such that* $\forall k > j_1$, $\mathbf{cx}(A^k,\,0) < \mathbf{cx}(A^\infty,\,0) + \delta$.

Proof. Let $\delta > 0$, $\mathbf{u}(A^\infty, 0)$ be any optimal solution for DLP(A^∞, 0), and let ϵ be any real number with

$$0 < \epsilon < \frac{\delta}{|\mathbf{u}(A^\infty,\,0)\mathbf{e}|}.$$

Assumptions 1 and 2 in the hypothesis ensure that Observation 1.1 and Lemma 2.1 are true. Hence, using the integer j_1 from Lemma 2.1, it follows that for $k > j_1$,

$$\begin{array}{llll} \mathbf{cx}(A^k,\,0) & \leq & \mathbf{cx}(A^\infty,\,\epsilon) & [\mathbf{x}(A^k,\,0) \text{ is feasible for LP}(A^\infty,\,\epsilon) \text{ by Lemma 2.1}] \\ & = & \mathbf{u}(A^\infty,\,\epsilon)\mathbf{b} + \epsilon\mathbf{u}(A^\infty,\,\epsilon)\mathbf{e} & \text{(duality theory)} \\ & \leq & \mathbf{u}(A^\infty,\,0)\mathbf{b} + \epsilon\mathbf{u}(A^\infty,\,0)\mathbf{e} & [\mathbf{u}(A^\infty,\,0) \text{ is feasible for DLP}(A^\infty,\,\epsilon) \text{ by Obs. 1.1}] \\ & = & \mathbf{cx}(A^\infty,\,0) + \epsilon\mathbf{u}(A^\infty,\,0)\mathbf{e} & \text{(duality theory)} \\ & < & \mathbf{cx}(A^\infty,\,0) + \delta & \text{(definition of } \epsilon) \end{array}$$

This establishes the desired inequality and completes the proof. □

2.2 Sufficient Conditions for the Inequality $\mathbf{cx}(A^k,\,0) > \mathbf{cx}(A^\infty,\,0) - \delta$.

The approach used to establish the second inequality requires that for every $\epsilon > 0$, the optimal solution to the limiting LP be feasible for the ϵ-outer LP of LP(A^k, 0), for all k sufficiently large, which is proved in the next lemma using Assumption 1.

Lemma 2.2 *If* $(A^k) \rightarrow A^\infty$ *and Assumption 1 holds, that is, for every* $k = 1, 2, \ldots$, $\mathbf{x}(A^k,\, 0)$ *is optimal for LP(*A^k*, 0) and* $\mathbf{x}(A^\infty,\, 0)$ *is optimal for LP(*A^∞*, 0), then* $\forall \epsilon > 0$, *there is an integer* j_2 *such that* $\forall k > j_2$, $\mathbf{x}(A^\infty,\, 0)$ *is feasible for LP(*A^k*,* ϵ*).*

Proof. Let $\epsilon > 0$ and note that if $A^k\mathbf{x}(A^\infty,\, 0) - A^\infty\mathbf{x}(A^\infty,\, 0) \leq \epsilon\mathbf{e}$, then $\mathbf{x}(A^\infty,\, 0)$ is feasible for LP(A^k, ϵ) because

$$\begin{aligned} A^k\mathbf{x}(A^\infty,\, 0) &\leq A^\infty\mathbf{x}(A^\infty,\, 0) + \epsilon\mathbf{e} \quad \text{(by assumption)} \\ &\leq \mathbf{b} + \epsilon\mathbf{e} \qquad\qquad [\mathbf{x}(A^\infty,\, 0) \text{ is feasible for LP}(A^\infty,\, 0)] \end{aligned}$$

It is now shown that there exists j_2 such that for all $k > j_2$, $A^k\mathbf{x}(A^\infty,\, 0) - A^\infty\mathbf{x}(A^\infty,\, 0) \leq \epsilon\mathbf{e}$, which will complete the proof.

Because $(A^k) \rightarrow A^\infty$, for each row $i = 1, \ldots, m$, $||A^k_{i.} - A^\infty_{i.}|| \rightarrow 0$. It follows that for each $i = 1, \ldots, m$, there is an integer k_i such that $||A^k_{i.} - A^\infty_{i.}|| \leq \epsilon / \|\mathbf{x}(A^\infty,\, 0)\|$ for all $k > k_i$. Let $j_2 = \max_i k_i$. Then for $k > j_2$ and for each row $i = 1, \ldots, m$, $||A^k_{i.} - A^\infty_{i.}|| \leq \epsilon / \|\mathbf{x}(A^\infty,\, 0)\|$ and so

$$\begin{aligned} A^k_{i.}\mathbf{x}(A^\infty,\, 0) - A^\infty_{i.}\mathbf{x}(A^\infty,\, 0) &= (A^k_{i.} - A^\infty_{i.})\mathbf{x}(A^\infty,\, 0) \\ &\leq ||A^k_{i.} - A^\infty_{i.}||\; ||\mathbf{x}(A^\infty,\, 0)|| \\ &\leq \epsilon \qquad\qquad \text{(by the choice of } j_2\text{)} \end{aligned}$$

Now i is arbitrary in the foregoing inequality, so $A^k\mathbf{x}(A^\infty,\, 0) - A^\infty\mathbf{x}(A^\infty,\, 0) \leq \epsilon\mathbf{e}$ for all $k > j_2$. As has already been shown, this implies that $A^k\mathbf{x}(A,\, 0) \leq \mathbf{b} + \epsilon\mathbf{e}$ and so $\mathbf{x}(A^\infty,\, 0)$ is feasible for LP(A^k, ϵ) and the proof is complete. □

A final assumption is needed to obtain the second inequality for the proof that the optimal objective function values of LP(A^k, 0) converge to that of LP(A^∞, 0). Two different such assumptions are provided here, the first of which is that there is a sequence of optimal solutions to DLP(A^k, 0) that lie inside a compact set, that is,

Assumption 3(a): There is a compact set $D \subset R^m$ and for each $k = 1, 2, \ldots$, an optimal solution $\mathbf{u}(A^k,\, 0)$ for DLP(A^k, 0) such that $\mathbf{u}(A^k,\, 0) \in D$.

It is now shown that the inequality in (2) holds under Assumptions 1 and 3(a).

Theorem 2.4 *If the sequence* $(A^k) \rightarrow A^\infty$ *and*

Assumption 1: *For every* $k = 1, 2, \ldots$, $\mathbf{x}(A^k,\, 0)$ *is optimal for LP(*A^k*, 0) and* $\mathbf{x}(A^\infty,\, 0)$ *is optimal for LP(*A^∞*, 0) and*

Assumption 3(a): *There is a compact set* $D \subset R^m$ *and for each* $k = 1, 2, \ldots$, *an optimal solution* $\mathbf{u}(A^k,\, 0)$ *for DLP(*A^k*, 0) such that* $\mathbf{u}(A^k,\, 0) \in D$,

then $\forall \delta > 0$, *there is an integer* j_2 *such that* $\forall k > j_2$, $\mathbf{cx}(A^k,\, 0) > \mathbf{cx}(A^\infty,\, 0) - \delta$.

Proof. Let $\delta > 0$ and, from Assumption 3(a), let $\mathbf{u}(A^k, 0)$ be an optimal solution for DLP$(A^k, 0)$ with $\mathbf{u}(A^k, 0) \in D$. Because D is compact, it follows that there is a real number $M > 0$ such that for each $k = 1, 2, \ldots,$

$$\mathbf{u}(A^k, 0)\mathbf{e} < M. \tag{8}$$

Now let ϵ be any real number with $0 < \epsilon < \delta/M$. Assumption 1 ensures that Observation 1.1 and Lemma 2.2 are true. Hence, using the integer j_2 from Lemma 2.2, it follows that for all $k > j_2$,

$$\begin{array}{rcll} c\mathbf{x}(A^\infty, 0) & \leq & c\mathbf{x}(A^k, \epsilon) & [\mathbf{x}(A^\infty, 0) \text{ is feasible for LP}(A^k, \epsilon) \text{ by Lemma 2.2}] \\ & = & \mathbf{u}(A^k, \epsilon)\mathbf{b} + \epsilon\mathbf{u}(A^k, \epsilon)\mathbf{e} & \text{(duality theory)} \\ & \leq & \mathbf{u}(A^k, 0)\mathbf{b} + \epsilon\mathbf{u}(A^k, 0)\mathbf{e} & [\mathbf{u}(A^k, 0) \text{ is feasible for DLP}(A^k, \epsilon) \text{ by Obs. 1.1}] \\ & < & \mathbf{u}(A^k, 0)\mathbf{b} + \epsilon M & [\text{from (8)}] \\ & = & \mathbf{cx}(A^k, 0) + \epsilon M & \text{(duality theory)} \\ & < & \mathbf{cx}(A^k, 0) + \delta & \text{(definition of } \epsilon) \end{array}$$

Equivalently,

$$c\mathbf{x}(A^k, 0) \quad > \quad c\mathbf{x}(A^\infty, 0) - \delta.$$

This establishes the desired inequality and completes the proof. □

In a specific application, it can be challenging to check Assumption 3(a). Using Assumptions 1 and 2, the next theorem provides a verifiable sufficient condition on the limiting LP—namely, that the set of optimal solutions to the dual of the limiting LP is compact—to ensure that Assumption 3(a) holds.

Theorem 2.5 *Suppose that Assumptions 1 and 2 hold. If there does not exist an m-vector $\mathbf{d} \neq \mathbf{0}$ such that $(A^\infty)^T\mathbf{d} = \mathbf{0}$, $\mathbf{d} \geq \mathbf{0}$, and $\mathbf{bd} = 0$, then there is a compact set $D \subset R^m$ and for each $k = 1, 2, \ldots,$ an optimal solution $\mathbf{u}(A^k, 0)$ for DLP$(A^k, 0)$ such that $\mathbf{u}(A^k, 0) \in D$.*

Proof. Suppose that the conclusion is not true. Thus, it is possible to construct a subsequence K of optimal solutions $\mathbf{u}(A^k, 0)$ for DLP$(A^k, 0)$ such that $\|\mathbf{u}(A^k, 0)\| \to \infty$ as $k \in K$. A contradiction is reached by showing that there is an m-vector $\mathbf{d} \neq \mathbf{0}$ such that $(A^\infty)^T\mathbf{d} = \mathbf{0}$, $\mathbf{d} \geq \mathbf{0}$, and $\mathbf{bd} = 0$. To that end, define the sequence

$$\mathbf{d}^k = \frac{\mathbf{u}(A^k, 0)}{||\mathbf{u}(A^k, 0)||} \quad \text{for } k \in K.$$

Clearly, $||\mathbf{d}^k|| = 1$ for all $k \in K$. Thus, there exists a subsequence K' of K and a vector $\mathbf{d} \in R^n$ with $\|\mathbf{d}\| = 1$ such that $(\mathbf{d}^k) \to \mathbf{d}$ as $k \in K'$. In addition, because $\mathbf{u}(A^k, 0) \geq \mathbf{0}$ for all $k \in K'$, $\mathbf{d}^k \geq \mathbf{0}$ for all $k \in K'$ and so $\mathbf{d} \geq \mathbf{0}$.

Now consider the sequence

$$(A^k)^T \mathbf{d}^k = \frac{(A^k)^T \mathbf{u}(A^k,\,0)}{\|\mathbf{u}(A^k,\,0)\|} = \frac{\mathbf{c}}{\|\mathbf{u}(A^k,\,0)\|} \quad \text{for } k \in K'.$$

Taking the limit over $k \in K'$ on both sides of the foregoing equality and noting that $\mathbf{c}$ is a constant vector and $\|\mathbf{u}(A^k,\,0)\| \to \infty$, it follows that $(A^\infty)^T \mathbf{d} = \mathbf{0}$.

Finally consider the sequence

$$\mathbf{b}\mathbf{d}^k = \frac{\mathbf{b}\mathbf{u}(A^k,\,0)}{\|\mathbf{u}(A^k,\,0)\|} = \frac{\mathbf{c}\mathbf{x}(A^k,\,0)}{\|\mathbf{u}(A^k,\,0)\|} \quad \text{for } k \in K'.$$

Taking the limit over $k \in K'$ on both sides of the foregoing equality and noting that, by Assumption 2, the values $\mathbf{x}(A^k,\,0)$ belong to a compact set and $\|\mathbf{u}(A^k,\,0)\| \to \infty$, it follows that $\mathbf{bd} = 0$.

That is, $\mathbf{d}$ satisfies $\mathbf{d} \neq \mathbf{0}$, $\mathbf{d} \geq \mathbf{0}$, $(A^\infty)^T \mathbf{d} = \mathbf{0}$, and $\mathbf{bd} = 0$. This contradicts the hypothesis and completes the proof. □

While Assumption 3(a) provides a sufficient condition on the collection of dual LPs to obtain the inequality in the conclusion of Theorem 2.4, the following assumption on the primal LPs provides another alternative.

Assumption 3(b). $\forall \epsilon > 0$, $\exists$ an integer k and a feasible solution $\bar{\mathbf{x}}^k$ for LP$(A^k,\,0)$ such that $\|\bar{\mathbf{x}}^k - \mathbf{x}(A^\infty,\,0)\| < \epsilon$.

The next theorem establishes the same inequality as in Theorem 2.4 except under Assumptions 1 and 3(b).

Theorem 2.6 *If the sequence* $(A^k) \to A^\infty$ *and*

Assumption 1: *For every* $k = 1, 2, \ldots,$ $\mathbf{x}(A^k,\,0)$ *is optimal for LP*$(A^k,\,0)$ *and* $\mathbf{x}(A^\infty,\,0)$ *is optimal for LP*$(A^\infty,\,0)$ *and*

Assumption 3(b): $\forall \epsilon > 0$, $\exists$ *an integer* k *and a feasible solution* $\bar{\mathbf{x}}^k$ *for LP*$(A^k,\,0)$ *such that* $\|\bar{\mathbf{x}}^k - \mathbf{x}(A^\infty,\,0)\| < \epsilon$,

then $\forall \delta > 0$, *there is an integer* j_2 *such that* $\forall k > j_2$, $\mathbf{c}\mathbf{x}(A^k,\,0) > \mathbf{c}\mathbf{x}(A^\infty,\,0) - \delta$.

Proof. Let $\delta > 0$. From Assumption 3(b), by setting $\epsilon = 1/k$ for each $k = 1, 2, \ldots,$ it is possible to construct a sequence $(\bar{\mathbf{x}}^k)$ of feasible solutions for LP$(A^k,\,0)$ such that $(\bar{\mathbf{x}}^k) \to \mathbf{x}(A^\infty,\,0)$. By continuity of the function $f(\mathbf{x}) = \mathbf{cx}$ at the point $\mathbf{x}(A^\infty,\,0)$, it follows that $(\mathbf{c}\bar{\mathbf{x}}^k) \to \mathbf{c}\mathbf{x}(A^\infty,\,0)$. In particular, for $\delta > 0$, $\exists$ an integer j_2 such that $\forall k > j_2$, $|\mathbf{c}\bar{\mathbf{x}}^k - \mathbf{c}\mathbf{x}(A^\infty,\,0)| < \delta$, so

$$\mathbf{c}\bar{\mathbf{x}}^k > \mathbf{c}\mathbf{x}(A^\infty,\,0) - \delta. \tag{9}$$

Thus, for $k > j_2$,

$$\begin{aligned} \mathbf{c}\mathbf{x}(A^k,\,0) &\geq \mathbf{c}\bar{\mathbf{x}}^k && (\bar{\mathbf{x}}^k \text{ is feasible for LP}(A^k,\,0)) \\ &> \mathbf{c}\mathbf{x}(A^\infty,\,0) - \delta && [\text{from } (9)] \end{aligned}$$

This establishes the desired inequality and completes the proof. □

It is interesting to note that Assumption 3(b) fails to hold in Example 2 in Section 1.1, thus explaining why the sequence of optimal objective function values in that example does not converge to the optimal objective function of the limiting problem. Also, Solow et al. (2001) use the special structure of the sequence of LPs in their application to establish that Assumption 3(b) holds. For other applications, it would be useful to have conditions on the sequence of LPs and the limiting LP under which Assumption 3(b) holds. One such condition is when $\mathbf{x}(A^\infty, 0)$ is feasible for each LP(A^k, 0), for then $\bar{\mathbf{x}}^k = \mathbf{x}(A^\infty, 0)$. Two additional conjectures that ensure the existence of the desired feasible solution $\bar{\mathbf{x}}^k$ are presented next.

Conjecture 2.1 *If whenever constraint i is binding at the optimal solution to LP(A^∞, 0), it follows that constraint i is binding at the optimal solution to LP(A^k, 0) for all k sufficiently large, then $\forall \epsilon > 0$, $\exists$ an integer k and a feasible solution $\bar{\mathbf{x}}^k$ for LP(A^k, 0) such that $\|\bar{\mathbf{x}}^k - \mathbf{x}(A^\infty, 0)\| < \epsilon$.*

Conjecture 2.2 *If $\exists$ an integer j such that $\forall k > j$, whenever constraint i is redundant for LP(A^k, 0), constraint i is redundant for LP(A^∞, 0), then $\forall \epsilon > 0$, $\exists$ an integer k and a feasible solution $\bar{\mathbf{x}}^k$ for LP(A^k, 0) such that $\|\bar{\mathbf{x}}^k - \mathbf{x}(A^\infty, 0)\| < \epsilon$.*

3 The Main Convergence Results

In this section, Assumptions 1, 2, and 3(a) and 3(b) are used to establish that the sequence of optimal objective function values of LP(A^k, 0) converge to the optimal objective function value of the limiting LP, even though the optimal solutions might not converge. As a consequence of this result, it is also shown that any convergent subsequence of optimal solutions to LP(A^k, 0) converges to an optimal solution of LP(A^∞, 0). A sufficient condition is also given under which the sequence of optimal solutions to the approximating LPs converges. Finally, results pertaining to convergent sequences of basic feasible solutions are presented.

Theorem 3.1 *If $(A^k) \to A^\infty$ and*

Assumption 1: *For every $k = 1, 2, \ldots$, $\mathbf{x}(A^k, 0)$ is optimal for LP(A^k, 0) and $\mathbf{x}(A^\infty, 0)$ is optimal for LP(A^∞, 0),*

Assumption 2: *There is a compact set C such that $(\mathbf{x}(A^k, 0)) \subseteq C$, and either*

Assumption 3(a): *There is a compact set $D \subset R^m$ and for each $k = 1, 2, \ldots$, an optimal solution $\mathbf{u}(A^k, 0)$ for DLP(A^k, 0) such that $\mathbf{u}(A^k, 0) \in D$, or*

Assumption 3(b) *$\forall \epsilon > 0$, $\exists$ an integer k and a feasible solution $\bar{\mathbf{x}}^k$ for LP(A^k, 0) such that $\|\bar{\mathbf{x}}^k - \mathbf{x}(A^\infty, 0)\| < \epsilon$,*

then $(\mathbf{cx}(A^k, 0)) \to \mathbf{cx}(A^\infty, 0)$, that is, $\forall \delta > 0$, there is an integer j such that $\forall k > j$, $|\mathbf{cx}(A^k, 0) - \mathbf{cx}(A^\infty, 0)| < \delta$.

Proof. Let $\delta > 0$. Assumptions (1) and (2) in the hypothesis ensure that the conclusion of Theorem 2.3 is true and so there is an integer j_1 such that for all $k > j_1$,

$$\mathbf{cx}(A^k,\, 0) < \mathbf{cx}(A^\infty,\, 0) + \delta.$$

For the other inequality, Assumption 3(a) or 3(b) ensures that the conclusion of either Theorem 2.4 or 2.6 is true. In either case, there is an integer j_2 such that for $k > j_2$,

$$c\mathbf{x}(A^k,\, 0) > c\mathbf{x}(A^\infty,\, 0) - \delta.$$

Letting $j = \max\{j_1, j_2\}$, the proof is completed by noting that for all $k > j$,

$$|\mathbf{cx}(A^k,\, 0) - \mathbf{cx}(A^\infty,\, 0)| < \delta. \quad \Box$$

In general, the sequence of optimal solutions for the approximating LPs need not converge, as shown in the following example.

Example 9: A Sequence of Optimal LPs that Converge to an Optimal LP in Which the Optimal Objective Function Values Converge but the Optimal Solutions Do Not Converge.

$\underline{\text{LP}^k \quad \text{(Optimal)}}$

$$\begin{array}{llllll} \max & x_1 & & & & \\ \text{s.t.} & x_1 & + & (-1)^k \frac{1}{k} x_2 & \le & 1 \\ & x_2 & & & \le & 1 \\ & x_1 & , & x_2 & \ge & 0 \end{array}$$

$$(x_1^k, x_2^k) = \begin{cases} (1 + \frac{1}{k}, 1), & \text{if } k \text{ is odd} \\ (1, 0), & \text{if } k \text{ is even} \end{cases}$$

$$x_1^k = \begin{cases} 1 + \frac{1}{k}, & \text{if } k \text{ is odd} \\ 1, & \text{if } k \text{ is even} \end{cases}$$

$\underline{\text{LP}^\infty \quad \text{(Optimal)}}$

$$\begin{array}{llllll} \max & x_1 & & & & \\ \text{s.t.} & x_1 & + & 0x_2 & \le & 1 \\ & x_2 & & & \le & 1 \\ & x_1 & , & x_2 & \ge & 0 \end{array}$$

$$(x_1^k, x_2^k) = (1, a) \quad \text{(where } 0 \le a \le 1)$$

$$x_1^k = 1$$

However, in the next theorem, it is shown that any convergent subsequence of $(\mathbf{x}(A^k,\, 0))$ converges to an optimal solution of the limiting LP.

Theorem 3.2 *If the assumptions in Theorem 3.1 hold, then any convergent subsequence of optimal solutions to LP(A^k, 0) converges to an optimal solution to LP(A^∞, 0).*

Proof. Let K be any subsequence such that $\mathbf{x}(A^k,\, 0)$ converges to $\mathbf{x}^\infty \in R^n$ as $k \in K$. It is clear that $\mathbf{x}^\infty$ is feasible for LP(A^∞, 0). The fact that $\mathbf{x}^\infty$ is optimal for LP(A^∞, 0) follows from Theorem 3.1 and continuity of the function **cx** because

$$\mathbf{cx}^\infty = \mathbf{c}[\lim_{k \in K} \mathbf{x}(A^k,\, 0)] = \lim_{k \in K} \mathbf{cx}(A^k,\, 0) = \mathbf{cx}(A^\infty,\, 0).$$

The proof is now complete. $\Box$

Although in general the sequence of optimal solutions to the approximating LPs need not converge, as seen in Example 9, if the optimal solution to the limiting LP is unique, then the sequence of optimal solutions to the approximating LPs does converge to the optimal solution of the limiting LP, as stated as a corollary to Theorem 3.2.

Corollary 3.1 *If the assumptions of Theorem 3.1 hold and* $\mathbf{x}(A^\infty, 0)$ *is the unique optimal solution to LP*$(A^\infty, 0)$*, then* $(\mathbf{x}(A^k, 0)) \to \mathbf{x}(A^\infty, 0)$ *as* $k \to \infty$.

In an actual application, the optimal solutions to the approximating problems are likely to be found by the simplex method and, as such, are going to be basic feasible solutions (bfs). A natural question to ask is whether any convergent subsequence of optimal bfs of the approximating LPs converges to an optimal bfs of the limiting problem. The following example shows that this need not always happen.

Example 10: A Sequence of Optimal LPs Whose Optimal bfs Do Not Converge to an Optimal bfs of the Limiting LP.

$$\begin{array}{llcccc}
\underline{\text{LP}^k \ \ (\text{Optimal})} & & & & & \\
\max & & & x_2 & & \\
\text{s.t.} & -\frac{1}{k}x_1 & + & x_2 & \leq & 1 \\
 & \frac{1}{k}x_1 & + & x_2 & \leq & 1 \\
 & x_1 & & & \leq & 1 \\
 & -x_1 & & & \leq & 1
\end{array}
\qquad
\begin{array}{llcccc}
\underline{\text{LP}^\infty \ \ (\text{Optimal})} & & & & & \\
\max & & & x_2 & & \\
\text{s.t.} & 0x_1 & + & x_2 & \leq & 1 \\
 & 0x_1 & + & x_2 & \leq & 1 \\
 & x_1 & & & \leq & 1 \\
 & -x_1 & & & \leq & 1
\end{array}$$

$$(x_1^k, x_2^k) = (0, 1), \quad x_2^k = 1 \qquad\qquad (x_1^\infty, x_2^\infty) = (0, 1) \text{ (not a bfs)}, \quad x_2^k = 1$$

One set of conditions ensuring that any convergent subsequence of optimal bfs of the approximating LPs converges to an optimal bfs of the limiting problem is provided in the next theorem.

Theorem 3.3 *If* $(A^k) \to A^\infty$ *and*

(a) For every $k = 1, 2, \ldots$, $\mathbf{x}(A^k, 0)$ *is an optimal bfs for LP*$(A^k, 0)$ *and,*

(b) For any sequence of nonsingular $(n \times n)$ *submatrices* (B^k) *of* (A^k) *in which each matrix* B^k *consists of the same rows of* A^k*, it follows that the limiting* $(n \times n)$ *submatrix* B^∞ *of* A^∞ *is nonsingular,*

then any convergent subsequence of $(\mathbf{x}(A^k, 0))$ *converges to an optimal bfs of LP*$(A^\infty, 0)$.

Proof. Let K be a subsequence and $\mathbf{x}^\infty \in R^n$ such that $(\mathbf{x}(A^k, 0)) \to \mathbf{x}^\infty$ as $k \in K$. Also, because each $\mathbf{x}(A^k, 0)$ is bfs for LP$(A^k, 0)$, let B^k be an $(n \times n)$ nonsingular submatrix of A^k such that, together with the remaining rows of A^k denoted by N^k,

$$B^k\mathbf{x}(A^k, 0) = \mathbf{b}_{B^k} \quad \text{and} \tag{10}$$

$$N^k\mathbf{x}(A^k, 0) \leq \mathbf{b}_{N^k}. \tag{11}$$

To establish that $\mathbf{x}^\infty$ is an optimal bfs for LP$(A^\infty, 0)$, it is first shown that $\mathbf{x}^\infty$ is a bfs for LP$(A^\infty, 0)$. To that end, a nonsingular $(n \times n)$ submatrix B^∞ of A^∞ is produced with the property that, together with the remaining rows of A^∞ denoted by N^∞,

$$B^\infty\mathbf{x}^\infty = \mathbf{b}_{B^\infty} \quad \text{and} \tag{12}$$

$$N^\infty\mathbf{x}^\infty \leq \mathbf{b}_{N^\infty}. \tag{13}$$

To construct this matrix B^∞, note that because there are a finite number of constraints, there is a subsequence K' of K such that for each $k \in K'$, the indices of the n binding constraints at the bfs $\mathbf{x}(A^k, 0)$ are all the same. Let B^∞ be the $(n \times n)$ submatrix of A^∞ corresponding to those binding constraints. Condition (b) in the hypothesis ensures that B^∞ is nonsingular. The fact that (12) and (13) hold follows by taking the limit on both sides of (10) and (11) over $k \in K'$. Thus, $\mathbf{x}^\infty$ is a bfs for LP(A^∞, 0).

It remains to show that $\mathbf{x}^\infty$ is optimal for LP(A^∞, 0). To that end, note that each $\mathbf{x}^k$ is an optimal bfs for LP(A^k, 0), so, for each $k \in K'$,

$$\mathbf{c}(B^k)^{-1} \geq \mathbf{0}. \tag{14}$$

Taking the limit over $k \in K'$ in (14), noting that (B^k) converges to the nonsingular matrix B^∞, it follows that the bfs $\mathbf{x}^\infty$ is optimal for LP(A^∞, 0) because

$$\mathbf{c}(B^\infty)^{-1} \geq \mathbf{0}.$$

The proof is now complete. □

Although no condition on the limiting LP has been found to ensure that hypothesis (b) in Theorem 3.3 holds, it is interesting to note that this hypothesis rules out the sequences of linear programming problems in the counter example in Example 10 in this section as well as those in Examples 2, 3, 5, and 7 in Section 1.1.

Summary

This paper provides results pertaining to solving a convergent sequence of linear programming (LP) problems. Specifically, given the data for a sequence of approximating LPs together with a limiting LP, the conditions in Theorem 3.1 in Section 3 are sufficient to ensure that (a) the sequence of optimal objective function values for the approximating problems converges to the optimal objective function value of the limiting problem (even though the optimal solutions might not converge) and (b) any convergent subsequence of optimal solutions to the approximating problems converges to an optimal solution of the limiting problem. In essence, these are sufficient conditions for the optimal objective function value of an LP to be a continuous function of the matrix of the LP. An area for future research is to identify sufficient conditions on the sequence of LPs and the limiting LP under which Assumption 3(b) in Theorem 3.1 in Section 2.3 holds—such as the open Conjectures 2.1 and 2.2 presented in Section 2.2.

The condition in Theorem 3.3 in Section 3.1—that there is no sequence of nonsingular submatrices consisting of the same set of rows of the matrices for the approximating problems that converges to a singular submatrix of the limiting problem—is sufficient to ensure that any convergent subsequence of optimal basic feasible solutions for the approximating problems converges to an optimal bfs for the limiting problem.

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