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**Future Capacity Procurements
under Unknown Demand and Increasing Costs**

by

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Future Capacity Procurements under Unknown Demand and Increasing Costs

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Abstract

In this paper we study a situation in which a broker must manage the procurement of a short life cycle product. As the broker observes demand for the item, she learns about the demand process. However, as is often the case in practice, it becomes either more difficult or more expensive to procure the item as the selling season advances. Thus, the broker must trade-off higher procurement costs against the benefit of making ordering decisions with better information about demand. Problems of this type arise, for example, in the travel industry, where a travel agent's cost of procuring airline and hotel reservations increases as the date of a vacation package approaches. We develop a newsvendor-like characterization of the optimal procurement policy. In a numerical analysis, we demonstrate how broker procurements tend to cluster just before price increases and how brokers can benefit from explicitly considering the effects of information about demand in their ordering policies.

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1. Introduction

In many supply chains, one (set of) firm(s) provides capacity to create products and a separate (set of) firm(s) performs market mediation. Market mediators typically obtain information about demand and attempt to obtain access to appropriate amounts of output from providers of capacity. Although these providers respond to requests from market mediators, they often encourage them to commit to orders in advance. Thus, from the perspective of the market mediators, it is as though they were brokers in a futures market for production capacity. In this paper, we examine a situation in which a market mediator accepts orders for delivery at a specific future date, and must obtain access to sufficient capacity to deliver as promised. Based on our analogy to a futures market for capacity, we generically refer to the capacity providers as *providers* and to the market mediators as *brokers* (of capacity).

The flexibility to order capacity incrementally over time is valuable to brokers when there is uncertainty about the demand, since it gives them the opportunity to postpone purchases until more information is available about demand. On the other hand, providers are usually keen to receive orders early, so that they obtain information about the demand and better plan their own activities. For this reason they may provide financial incentives such as discounts to the brokers to induce them to commit early. Therefore, the desire to manage the demand risk results in conflicting objectives for the brokers and the providers. Consider for example, a travel agent selling a certain vacation package. Because of uncertainty as to the amount of demand for the package, the travel agent would like to delay booking reservations with airlines, hotels, etc. in order to observe as much of the demand for the package as possible. On the other hand, because the yield management practices of hotels and airlines tend to result in discounts for early commitments, the travel agent may have to pay more for the reservations if she delays.

This conflict is important especially when a new product is introduced, since there is often a considerable amount of uncertainty as to the rate at which demand will occur. The uncertainty decreases as sales data is observed and firms *learn* about demand. Therefore, the longer a firm can delay making procurement commitments, the more it will know about demand when such commitments are made. On the other hand, there are two main reasons why capacity providers offer discounts to brokers who commit early. First, the discounts encourage speculative purchases, stimulating demand among brokers who are less sensitive to risk. Second, since the orders contain information about final demand, the discounts are a mechanism that allow the providers to obtain information sooner than they would otherwise. With better information about its own demand, a provider may be better able to plan its own activities and schedule the availability of various factors of production. As the selling season advances, it becomes increasingly difficult to make short term re-allocations of the factors of production and or deliver rush orders to the brokers. According to a recent article in “The Wall Street Journal” (see Bird and Bounds 1997), having to rush last minute orders for the Christmas season by air can be five times more expensive than shipping early.

In such settings where financial incentives encourage early commitments, capacity brokers are faced with an important tradeoff: as they delay the placement of orders, they gain information about demand, but have to pay higher procurement costs. This tradeoff is the primary focus of this paper. We analyze the broker's problem as a dynamic program and show that the optimal procurement policy can be characterized by order-up-to thresholds that depend upon the number of observed demands. In a numerical study we examine how the specific path of the cost function affects the form of the broker's threshold policy, as well as the benefit to the broker by following an adaptive procurement policy instead of myopically ordering at the start of the selling season and ignoring the opportunity to learn about demand.

The remainder of the paper is organized as follows. In Section 2, we review the relevant literature. In Section 3, we develop and analyze a model of the problem for increasing procurement cost functions and characterize an optimal policy. In Section 4, we develop newsvendor-like bounds on the optimal ordering thresholds. These bounds reduce to the optimal thresholds when there is only one price increase remaining in the horizon. In Section 5, we describe and discuss some computational results for specific cost functions that are of particular interest. Finally, we summarize our results in Section 6.

2. Literature Review

The literature on inventories for short life-cycle products typically addresses one of the following two issues: the allocation of a given amount of inventory to overlapping classes of demand, or the use of price as a short-term decision variable. Much of the yield management literature studies the problem of allocating a fixed number of seats, hotel rooms, etc. to a set of overlapping classes of demand in order to maximize revenue. See Belobaba (1987) for a review of yield management. Another set of the literature studies the problem of how a firm should dynamically adjust the price of a product that can be sold only during a fixed amount of time. Examples of this type of research include Besanko and Winston (1990), Bitran and Mondschein (1994) and Gallego and van Ryzin (1994). In contrast to the situation that we address, these papers assume that the price can be adjusted dynamically, while the stocking policy is determined exogenously. Indeed, our paper can be viewed as complementary to the yield management literature in the sense that we explicitly model how a customer firm will optimally respond if its supplier is increasing prices, either using yield management techniques or otherwise, over time. Thus, part of the contribution of our paper is to provide insight into how demand will respond to yield management techniques.

Since our paper specifically addresses the issue of learning about demand, it is related to research on the effects of incomplete information about the demand distribution of a short life-cycle product. At least two authors have addressed the problem of making ordering and/or pricing decisions for perishable products for which the parameters of the demand distribution are uncertain. Lariviere and Porteus (1999) model the problem that a broker faces in placing a series of orders for a perishable product. Because the

product is perishable, this is equivalent to a sequence of newsvendor problems, where the broker's information about the demand distribution is changing from one problem to the next. Using a Bayesian framework to model the broker's *learning* about demand, they examine the way in which the broker's decisions are influenced by the wholesale price schedule that is offered by the supplier of the product. They also model the wholesaler's problem of setting the price schedule using a game theoretic approach. Burnetas and Smith (2000) also study the problem of ordering in a series of newsvendor problems under an unknown demand distribution. Their work differs from Lariviere and Porteus' in two respects: First, they make no distinction between the interests of the broker and those of the manufacturer, and second they make no parametric assumptions about the distribution of demand. They demonstrate that a simple adaptive ordering policy converges to the newsvendor solution that would be optimal if the demand distribution were known. In the context of inventory policies for non-perishable products under incomplete information on demand, Azoury (1985) establishes a class of distributions for which there exist sufficient statistics and Bayes conjugate priors. Lovejoy (1990) shows that a class of simple myopic policies can be optimal or near-optimal in several inventory models with uncertain demand distributions.

The problem we study is perhaps most similar to the one described and modeled by Murray and Silver (1966). However, our approach differs from theirs in several significant ways. First, we assume that the broker can order at any period instead of only at a set of externally imposed points in time. Second, the emphasis of Murray and Silver's work is in the development of computational short-cuts for solving the dynamic program, while we address the structural properties of the optimal ordering policy.

3. The Model

Consider a broker who faces uncertain demand and increasing wholesale costs for a product that is sold over a finite horizon of N discrete time periods. We assume that the broker sells the product for a fixed price p , and that at the end of the horizon she receives a salvage value of s per unit of excess stock. The assumption of a fixed selling price can be justified in environments such as catalogue sales where retailers often advertise and commit to a specific selling price, eliminating much of their flexibility to change it dynamically. Furthermore, by fixing the selling price, we focus our attention on how wholesale cost increases affect procurement.

The unit wholesale cost to the broker for orders placed at the beginning of period j is denoted by c_j , where c_j is non-decreasing in j , and $c_1 \geq s$. For simplicity, we assume that there are no fixed costs or capacity constraints associated with the placement of orders, and that the amount ordered is instantaneously available to the broker. These assumptions are reasonable for products, such as travel packages, for which sales are made prior to a specific date upon which all associated services are performed and/or physical products are exchanged. They also may apply to certain seasonal products for which the broker guarantees delivery by a specific expiration date, such as a popular toy that must be delivered to customers prior to Christmas Day.

Of course in more traditional retail environments setup costs must be taken into account when considering procurement policies. However including them in the present model would tend to lump orders together, thus disguising the tradeoff between increasing procurement cost and learning about demand. Since the focus of this paper is on the latter tradeoff, we do not consider setup costs.

If a demand arrives during period j and the stock is at zero level, then there is a profit equal to r_j associated with this unit of demand. By varying r_j , it is possible to model different ways of dealing with out-of-stock situations. For example, $r_j = 0$ corresponds to lost sales. Alternatively, in cases such as the travel agent example, if a request occurs while no inventory is on hand, it can be satisfied by instantaneously ordering one item at the price c_{j+1} that will be available at the beginning of period $j+1$, perhaps with an additional backlog penalty. For this case we would set $r_j = p - c_{j+1} - B_j$, for some $B_j \geq 0$. Finally, $r_j = \max(p - c_{j+1} - B_j, 0)$ represents the case where the broker has the option not to satisfy the unit of demand that occurred while out of stock, if it is not profitable to do so. In this paper we assume that r_j satisfies the following property

$$r_j \leq \text{Max}\{p - c_{j+1}, r_{j+1}\} \leq p - s, \quad j = 1, 2, \dots, N \quad (1)$$

where c_{N+1} denotes the unit ordering cost after the N^{th} scheduled reorder time has expired. Because $p - c_{j+1}$ is nonincreasing in j , an equivalent way to state (1) is that, for any j , either $r_j \leq p - c_{j+1}$, or $r_j \leq r_k \leq p - s$, for all $k = j+1, \dots, N$.

Note that all three examples presented above satisfy this property. The implication of (1) is that the N -period horizon can be divided into two parts $1, \dots, j_0$ and j_0+1, \dots, N , such that r_j is below the net profit $p - c_{j+1}$ but otherwise arbitrary in the earlier part and above $p - c_{j+1}$ and non-decreasing in the latter. In practice it is usually the case that $r_j \leq p - c_{j+1}$, because of backlog penalties associated with stockouts. However it is also possible for the opposite inequality to hold, as it happens, for example, if $p < c_{j+1}$ for $j \geq j_0$ and $r_j = \max(p - c_{j+1}, 0)$, in which case r_j is constant (therefore non-decreasing) during the second subinterval. Although it is unlikely for r_j to be strictly increasing in practical situations, all results derived in this section also hold for this more general case.

We use a discrete time framework to model the demand arrival process and we assume that the demand X_j in period j is a Bernoulli random variable with success probability λ . Since the length of a period can be considered to be sufficiently small, our model is most appropriate for situations in which demand arises from many small independent customers. For simplicity we assume that parameter λ is time-invariant, but all of our results can be generalized for the case in which $\lambda_j = d_j \lambda$, where d_j is a deterministic time varying multiplier.

We use a Bayesian model for the broker's learning mechanism. Specifically we assume that the broker does not know the true value of λ . She instead has a prior distribution for λ that is Beta with parameters α and β . Although our choice of the Beta prior is driven by the fact that it is the conjugate prior

for the Bernoulli distribution, it should be noted that it is extremely flexible in modeling a range of values for the prior mean and variance of λ . Hence this assumption is not very restrictive.

In each time period the broker observes whether a unit of demand has occurred or not and she updates the distribution of λ appropriately. Specifically, if at the end of period j the broker has observed n units of demand, then the posterior distribution for λ in the beginning of period $j+1$ is also Beta with parameters $\alpha + n$ and $\beta + j - n$. Let $g_j(\lambda, n)$ denote the posterior density function of λ in the beginning of period $j+1$, given that n units of demand have occurred by the end of period j :

$$g_j(\lambda, n) = \frac{\lambda^{\alpha+n} (1-\lambda)^{\beta+j-n}}{B(\alpha+n, \beta+j-n)}, \quad \lambda \in [0,1] \quad (2)$$

where: $B(\alpha+n, \beta+j-n) = \frac{\Gamma(\alpha+n)\Gamma(\beta+j-n)}{\Gamma(\alpha+\beta+j)}$, and $\Gamma(k) = (k-1)!$ for integer k .

To model the optimization problem, we assume that in each period $j+1$ the broker first observes the stock level x_{j+1} then orders (and immediately receives) $y_{j+1} - x_{j+1}$ units, where $y_{j+1} \geq x_{j+1}$, at a cost of c_{j+1} per unit, and finally experiences the demand during period $j+1$.

The broker's problem can now be represented by the following dynamic program. Let $v_{j+1}(n, x)$ be the optimal profit during periods between $j+1, \dots, N$, given that n units of demand have been observed during periods $1, \dots, j$ and that there are x units at hand. Then, v_{j+1} satisfies the following recursion

$$v_{j+1}(n, x) = c_{j+1}x + \text{Max}_{y \geq x} \{w_{j+1}(n, y)\}, \quad j = 0, \dots, N-1 \quad (3a)$$

$$v_{N+1}(n, x) = s x, \quad (3b)$$

where, for $y > 0$,

$$w_{j+1}(n, y) = -c_{j+1}y + \int_0^1 g_j(\lambda, n) (\lambda(p + v_{j+2}(n+1, y-1)) + (1-\lambda)v_{j+2}(n, y)) d\lambda \quad (4a)$$

and for $y=0$,

$$w_{j+1}(n, 0) = \int_0^1 g_j(\lambda, n) (\lambda(r_{j+1} + v_{j+2}(n+1, 0)) + (1-\lambda)v_{j+2}(n, 0)) d\lambda. \quad (4b)$$

Note that the conditional expectation of λ given n and j is equal to

$$\int_0^1 g_j(\lambda, n) \lambda d\lambda = \frac{\alpha+n}{\alpha+\beta+j}.$$

Substituting this into (4), it follows after some algebra that, for $y > 0$:

$$w_{j+1}(n, y) = -c_{j+1}y + \frac{\alpha + n}{\alpha + \beta + j} (p + v_{j+2}(n+1, y-1)) + \left(1 - \frac{\alpha + n}{\alpha + \beta + j}\right) v_{j+2}(n, y) \quad (5a)$$

and

$$w_{j+1}(n, 0) = \frac{\alpha + n}{\alpha + \beta + j} (r_{j+1} + v_{j+2}(n+1, 0)) + \left(1 - \frac{\alpha + n}{\alpha + \beta + j}\right) v_{j+2}(n, 0) . \quad (5b)$$

In order to establish that the optimal ordering policy is a threshold policy, we need to establish monotonicity and concavity properties of the value function. We will do this by using the following first order differences. Let us define:

$$\delta_{j+1}(n, y) = w_{j+1}(n, y+1) - w_{j+1}(n, y) \quad (6)$$

$$d_{j+1}(n, x) = v_{j+1}(n, x+1) - v_{j+1}(n, x) . \quad (7)$$

Using (3) and (5) we see that these differences are related as follows.

$$\delta_{j+1}(n, y) = -c_{j+1} + \frac{\alpha + n}{\alpha + \beta + j} d_{j+2}(n+1, y-1) + \left(1 - \frac{\alpha + n}{\alpha + \beta + j}\right) d_{j+2}(n, y) \quad (8a)$$

for $y > 0$, and

$$\delta_{j+1}(n, 0) = -c_{j+1} + \frac{\alpha + n}{\alpha + \beta + j} (p - r_{j+1}) + \left(1 - \frac{\alpha + n}{\alpha + \beta + j}\right) d_{j+2}(n, 0) . \quad (8b)$$

The above first order differences can be used to confirm that the broker's expected optimal profit is non-decreasing in the size of her initial inventory level. This is intuitive: since the cost of procuring the initial inventory is not captured by the value function, each additional unit of initial inventory can potentially either reduce the number of units procured in the future or increase the number of units sold. The following proposition formalizes this intuition.

Proposition 3.1: The optimal profit $v_{j+1}(n, x)$ is non-decreasing in the stock level x .

A proof of this result is provided in the appendix. We are now ready to characterize the structure of an optimal ordering policy. The following theorem establishes that there exists an optimal procurement policy of threshold type.

Theorem 3.1:

- a) The following inequality holds : $d_{j+1}(n, 0) \leq p - r_j$, for all j, n .
- b) $w_{j+1}(n, y)$ is concave in y for all $j \leq N-1$, and $v_{j+1}(n, y)$ is concave in y for all $j \leq N$.

c) For all $j \leq N-1$ and all n there exist thresholds

$$y_{j+1}^l = \text{Max}\{y : \delta_{j+1}(n, y-1) > 0\}, \text{ where Max } \emptyset = 0, \quad (9a)$$

$$y_{j+1}^u = \text{Min}\{y : \delta_{j+1}(n, y) < 0\}, \quad (9b)$$

such that an optimal policy in period $j+1$ and state (n, x) is to stock a quantity equal to $\max(x, y)$, where y is any stock level $y_{j+1}^l \leq y \leq y_{j+1}^u$.

Proof: We will show all three parts by simultaneous backward induction on j . Since, by (5), $w_{j+1}(n, y)$ is a convex combination of terms involving function v_{j+2} , we only need to include the concavity of v in the induction hypothesis, since that implies the concavity of w .

We first consider the case $j=N$. Note that, by (1) and (8b), $d_{N+1}(n, 0) = s < p - r_N$, and $v_{N+1}(n, y) = sy$ is linear, thus concave. In addition, it is obvious that $y_{N+1}(n) = 0$, for all n , i.e. with no further opportunity for sales, the optimal stocking level is zero, therefore part (c) holds. We now make the inductive hypothesis that, for some $j \leq N-1$, $d_{j+2}(n, y)$ is non-increasing in y , and $d_{j+2}(n, 0) \leq p - r_{j+1}$, for all n . From (8a) it follows that the first order difference $\delta_{j+1}(n, y)$ is non-increasing in y , for $y > 0$. To prove this for $y=0$ as well, it follows from (8a) and (8b) that:

$$\delta_{j+1}(n, 1) - \delta_{j+1}(n, 0) = \frac{\alpha + n}{\alpha + \beta + j} (d_{j+2}(n+1, 0) - p + r_{j+1}) + \left(1 - \frac{\alpha + n}{\alpha + \beta + j}\right) (d_{j+2}(n, 1) - d_{j+2}(n, 0)) \leq 0,$$

where the inequality follows from the induction hypothesis. Therefore $\delta_{j+1}(n, y)$ is non-increasing in y for all y and $w_{j+1}(n, y)$ is concave in y . Hence, there exists a range of values of y , as defined in (9), for which $w_{j+1}(n, y)$ is maximized. Therefore, if the inventory at the beginning of period $j+1$ is equal to x , it is optimal to be replenished up to y , if $x < y$, while no replenishment takes place if $x \geq y$, where y is any number between y_{j+1}^l and y_{j+1}^u . Note any such y corresponds to an optimal stocking level at $j+1$ when n units have been sold, and there is no on-hand inventory, i.e. $x=0$. This proves part (c).

Using the form of the optimal policy, the value function can be expressed as

$$v_{j+1}(n, x) = \begin{cases} c_{j+1}x + w_{j+1}(n, y_{j+1}(n)), & \text{if } x < y_{j+1}^l(n) \\ c_{j+1}x + w_{j+1}(n, x), & \text{if } x \geq y_{j+1}^l(n) \end{cases}$$

From the definition of $y_{j+1}^l(n)$ it follows that $\delta_{j+1}(n, y) > 0$, if and only if $y < y_{j+1}^l(n)$. Therefore,

$$d_{j+1}(n, x) = c_{j+1} + \text{Min}\{0, \delta_{j+1}(n, x)\}. \quad (10)$$

Since $\delta_{j+1}(n,y)$ is non-increasing in y , $d_{j+1}(n,y)$ is also non-increasing, thus $v_{j+1}(n,y)$ is concave in y .

To complete the proof, it remains to show that $d_{j+1}(n,0) \leq p-r_j$. By the induction hypothesis, $d_{j+2}(n,0) \leq p-r_{j+1}$, therefore, from (8b), $\delta_{j+1}(n,0) \leq -c_{j+1}+p-r_{j+1}$, and from (10),

$$d_{j+1}(n,0) \leq c_{j+1} + \text{Min}\{0, -c_{j+1}+p-r_{j+1}\} = \text{Min}\{c_{j+1}, p-r_{j+1}\}.$$

There are two possible cases for r_j :

If $r_j \leq p-c_{j+1}$, then $c_{j+1} \leq p-r_j$. If, on the other hand, $r_j > p-c_{j+1}$, then, by (2), $r_j \leq r_{j+1}$, thus $p-r_{j+1} \leq p-r_j$. In both cases, $d_{j+1}(n,0) \leq \text{Min}\{c_{j+1}, p-r_{j+1}\} \leq p-r_j$, and the induction proof is complete. \blacklozenge

We now investigate how the broker's stocking problem is affected by its parameters. In the following proposition we establish the property that the marginal benefit from an additional unit of inventory is higher when a larger number of demand units has been observed. Note that, as defined in (6) and (7), $\delta_{j+1}(n,y)$ can be interpreted as the marginal profit from the procurement of an additional unit, and $d_{j+1}(n,x)$ as the marginal profit from an additional unit of initial inventory.

Proposition 3.2: The differences $\delta_{j+1}(n,y)$, $d_{j+1}(n,y)$ are non-decreasing in n , for all $j \leq N$ and all y .

This result, for which a proof is provided in the appendix, has the following practical implication:

Corollary 3.1: The optimal stocking level limits $y_{j+1}^l(n)$ and $y_{j+1}^u(n)$ are non-decreasing in n for all j .

The corollary follows immediately from the preceding proposition and the definitions of $y_{j+1}^l(n)$ and $y_{j+1}^u(n)$ in (9).

The preceding results confirm the intuition that, at any given point in time, the more units (n) of demand that have been observed, the larger the order-up-to level. It is also of interest to investigate how the order-up-to level is affected when time passes without additional demand being observed. The following proposition addresses precisely this issue.

Proposition 3.3: $y_{j+1}^u(n)$ is non-increasing in j for all n .

The above result, a proof of which is given in the appendix, is important because it implies that there exists an optimal restocking policy in which orders may be placed only at the end of periods in which non-zero demand occurs. Thus, a broker can optimally manage her procurement costs by evaluating procurement decisions only at instances when demand occurs.

Remark 3.1: The analysis in this section was based on the assumption that stockouts are handled either as lost sales, or through the immediate procurement of one unit of product, perhaps with a backlog penalty. An alternative way to model this situation is to require the broker to maintain a base stock of at least some

minimum amount of inventory. Although a base stock requirement cannot be recovered as a special case of our original model by assuming appropriate values of r_j , the analysis of a model with a base stock requirement is very similar to what we have presented here. Specifically, since zero stock levels are not allowed, only the expressions for $x > 0$ and $y > 0$ in the value function and first order differences are applicable. Furthermore, the maximization in the dynamic programming equations (3a) is taken over the range $y \geq x$ and $y \geq x_{bs}$, where x_{bs} is the minimum base stock requirement. All results developed in this section regarding the structure and properties of the optimal policy and value function hold unchanged.

Remark 3.2: Consider the special case of the general model, where, either by setting the parameter values appropriately, or by explicitly requiring it, all the arriving demand must be satisfied. In this case the expected revenue from selling the product is independent of the procurement policy. Therefore, the problem of maximizing the expected profit is equivalent to minimizing the expected procurement cost minus the salvage value of the ending inventory, and the optimal procurement policy is independent of the retail price p . Two common cases where the demand is always satisfied are the original model with $r_j = p - c_{j+1}$, or the modified model discussed in Remark 3.1.

4. Bounds on the Broker's Optimal Ordering Policy

With the preceding results, we are now in a position to explore the manner in which the broker should optimally respond to specific cost functions. Due to their prevalence in practice, we are primarily interested in cost functions, hereafter referred to as step-up, in which the set of periods in which there is an increase is a small subset of all of the periods. (Recall that the analysis in Section 3 allows for an arbitrary number of cost increases). To simplify the presentation, we describe the details of the analysis for the case in which all demand is satisfied (either by procuring an item immediately for any demand that occurs during a stockout, or by maintaining a basestock inventory of at least one unit as described in Remark 3.1). However, we also outline parallel results for the cases where demands occurring during a stockout are either lost, or the broker has the option of satisfying them by procuring at the soonest opportunity.

We first consider a situation in which a single “pre-season” discount is available prior to some pre-specified date. As a result, the broker faces a cost function as shown in Figure 1. Let c_d be the discounted wholesale price that is available up to and including some period, to which we will refer as j_1 , and c_f the full wholesale price after period j_1 . As shown in the following theorem, this simple cost structure results in a simple, intuitively appealing solution.

Theorem 4.1: Suppose that $r_j = p - c_{j+1}$ and the procurement cost c_j can be represented as $c_j = c_d$ for $j \leq j_1$, and $c_j = c_f$ for $j > j_1$ for some $c_f > c_d \geq 0$ and $j_1 \in \{1, 2, \dots, N\}$. Then, the optimal stocking level limits can be characterized as follows:

$$y_j^l(n) = 0, \quad y_j^u(n) \geq y_{j_1}^u(n), \quad \text{for } j < j_1, \quad (12a)$$

$$y_{j_1}^l = y_{j_1}^u = H\left(j_1 - 1, n; \frac{c_f - c_d}{c_f - s}\right), \quad (12b)$$

$$y_j^l(n) = y_j^u(n) = 0, \text{ for } j > j_1, \quad (12c)$$

where

$$H(j, n; R) = \text{Min} \{y : F(j, n; y) > R\}, \quad (12d)$$

$$F(j_1, n; x) = \sum_{y=0}^x K(y) \frac{\Gamma(\alpha + n + y)\Gamma(\beta + N - n - y)}{\Gamma(\alpha + n)\Gamma(\beta + j_1 - n)} \text{ for } x \in \{0, \dots, N - j_1\}, \quad (12e)$$

and

$$K(y) = \frac{\Gamma(\alpha + \beta + j_1)(N - j_1)!}{\Gamma(\alpha + \beta + N)(N - j_1 - y)!y!}. \quad (12f)$$

Proof: The limits for $j > j_1$ in (12d) or (12e) are obviously optimal for the corresponding case, since there is no advantage in purchasing in advance of demand arrivals after the discount expires at the end of period j_1 . To show (12a) for $1 \leq j < j_1$, we use (8b) together with $p - r_j = c_{j+1} = c_j = c_d$ and $d_{j+1}(n, 0) \leq p - r_{j-1} = c_d$, to obtain

$$\delta_j(n, 0) = \left(1 - \frac{\alpha + n}{\beta + j - 1 + n}\right)(d_{j+1}(n, 0) - c_d) \leq 0.$$

Therefore, from (9a), $y_j^l(n) = 0$. The inequality in (12a) follows from Proposition 3.3.

We finally show (12b). Note that, since $r_j = p - c_{j+1}$, which implies that all arriving demands are satisfied at the current cost, at the beginning of period j_1 the broker faces a newsvendor problem in which the understocking cost is $c_f - c_d$, and the overstocking cost is $c_d - s$. In addition, if n units of demand have occurred by the end of some period j , then the probability that no more than x units of demand occur by the end of the horizon can be expressed as:

$$F(j, n; x) = \int_{\lambda=0}^{\infty} g_j(\lambda, n) \sum_{y=0}^x \binom{N-j}{y} \lambda^y (1-\lambda)^{N-j-y} \text{ for } x \in \{0, \dots, N-j\}$$

This expression follows from the fact that, for a given value of λ , the conditional density for demand during the remaining $N-j$ periods is binomial with parameters λ and $N-j$. By substituting $g_j(\lambda, n)$ from (1) and re-arranging terms, it is easy to show that:

$$F(j_1, n; x) = \int_{\lambda=0}^{\infty} \frac{\Gamma(\alpha + \beta + j_1)}{\Gamma(\alpha + n)\Gamma(\beta + j_1 - n)} \lambda^{\alpha+n-1} (1-\lambda)^{\beta+j_1-n-1} \sum_{y=0}^x \binom{N-j_1}{y} \lambda^y (1-\lambda)^{N-j_1-y}$$

$$\begin{aligned}
&= \frac{\Gamma(\alpha + \beta + j_1)(N - j_1)}{\Gamma(\alpha + n)\Gamma(\beta + j_1 - n)\Gamma(\alpha + \beta + N)} \sum_{y=0}^x \frac{\Gamma(\alpha + n + y)\Gamma(\beta + N - n - y)}{(N - j_1 - y) y!} \\
&\quad \int_{\lambda=0}^{\infty} \frac{\Gamma(\alpha + \beta + N)}{\Gamma(\alpha + n + y)\Gamma(\beta + N - n - y)} \lambda^{\alpha+n+y-1} (1 - \lambda)^{\beta+N-n-y-1} d\lambda \tag{13}
\end{aligned}$$

where $\Gamma(n) = (n-1)!$ for integer values of n . We can now recognize the latter integrand as a beta probability density function with parameters $\alpha+n+y$ and $\beta+N-n-y$. Thus the integral is equal to one. After removing this term from (13), simple algebraic manipulations demonstrate that it is equal to (12d). Now (12b) follows as the unique optimal ordering policy of this newsvendor problem. \blacklozenge

As a consequence of Theorem 4.1, one optimal ordering policy for this case of cost function is to stock up to level $y_{j_1}^u(n)$, in periods $1, \dots, j_1$, and order nothing in periods j_1+1, \dots, N . The intuition behind this result is quite simple. At any time $j \leq j_1$, we can anticipate the distribution of demand at the point of the price increase conditional on there being no additional demands between time j and j_1 . By solving the newsvendor problem for this distribution, we know exactly how much we would want to have on hand in period j_1 in the event that no more demands occur. If, on the other hand, additional demands do occur prior to j_1 , this can only increase the amount that we would like to have on hand at j_1 . Thus, (12b) represents a lower bound on what the optimal stocking quantity will be at time j_1 , for any realization of demand between j and j_1 .

Theorem 4.1 is also in agreement with the discussion in Remark 3.2, since, by assuming that the demand is always satisfied, the optimal procurement policy is independent of the retail price p .

We can use this intuition to examine the general case for step-up cost functions, such as the one shown in Figure 2, in which the cost increases at z discrete points in time. Let j_i be the last period before the i^{th} increase in price occurs, for $i = 1, \dots, z$, and define c^1 to be the initial cost. The step-up cost function can be expressed formally as:

$$c_j = c^i \text{ for } j \in \{j_{i-1}+1, \dots, j_i\} \tag{15}$$

where $c^i \leq c^{i+1}$ for $i = 1, \dots, z$.

Theorem 4.2: If $r_j = p - c_{j+1}$ and the procurement cost is given by the step-up form in (15), then any optimal stocking level y at any period $j < j_z$ is bounded in the following way:

$$H\left(j_k, n; \frac{c^{k+1} - c^k}{c^{k+1} - s}\right) \leq y \leq H\left(j_k, n; \frac{c^{z+1} - c^k}{c^{z+1} - s}\right), \tag{16}$$

where $k = \text{Min}\{i: j_i \geq j\}$, H is as defined in (12d) and

$$F(j_k, n; x) = \sum_{y=0}^x K(y, j_k) \frac{\Gamma(\alpha + n - y) \Gamma(\beta + N - n - y)}{\Gamma(\alpha + n) \Gamma(\beta + j_k - n)} \quad \text{for } x \in \{0, \dots, N - j_k\}, \quad (17a)$$

$$K(y, j_k) = \frac{\Gamma(\alpha + \beta + j_k) (N - j_k)!}{\Gamma(\alpha + \beta + N) (N - j_k - y)! y!}, \quad (17b)$$

Proof: The fact that (17) is the cumulative distribution of demand for periods j_{k+1} through N given that exactly n demands have occurred by period j_k , can be shown using the same approach as was used in the proof of Theorem 4.1. We first show the left-hand inequality in (16). The per unit cost of any shortage that occurs after time j_k is at least $c_{k+1} - c_k$, since we will have to acquire the item at a cost higher than that available at time j_k . Therefore, a lower bound for the optimal stocking level at time j_k is the solution of a newsboy problem with understocking cost equal to $c_{k+1} - c_k$ and overstocking cost equal to $c_k - s$. To show the right-hand inequality, note that the per unit cost of any shortage that occurs after period j_k is no more than $c_{z+1} - c_k$, since the opportunity cost of delaying the acquisition of a unit from j_k until any period $j > j_k$ is no more than $c_{z+1} - c_k$. \blacklozenge

The tightness of these bounds depends upon the size of the gap between c_k and c_{z+1} . Also note that in any period $j \in \{j_{z-1}+1, \dots, j_z\}$, both the upper and lower bounds in (16) collapse to the optimal ordering policy. Thus, by setting $z = 1$ and $c^0 = c_d, c^1 = c_f$, Theorem 4.1 becomes a special case of Theorem 4.2.

Remark 4.1: Theorems 4.1 and 4.2 were developed under the assumption that $r_j = p - c_{j+1}$. For the case where there is a requirement for a base stock of one unit, they are still true with $y_j^l(n) = 1$, for $j \neq j_1$. Furthermore, we have developed parallel results for situations where either sales are lost, or the broker has the option to satisfy a request occurring after a stockout with an instantaneous procurement. These situations correspond to $r_j = 0$ and $r_j = \text{Max}\{p - c_j, 0\}$, respectively. For either of these cases, a lower bound for the optimal stocking level in period j_k can be obtained by viewing the stocking decision at time j_k as a newsvendor problem with understocking cost equal to $\text{Max}\{p - c_k - r_{k+1}, 0\}$ and overstocking cost equal to $c_k - s$. The critical ratio for this newsvendor problem is equal to

$$\frac{\text{Max}\{p - c_k - r_{k+1}, 0\}}{\text{Max}\{p - c_k - r_{k+1}, 0\} + c_k - s} = \text{Max}\left\{\frac{p - c_k - r_{k+1}}{p - r_{k+1} - s}, 0\right\}. \quad (18)$$

Conversely, an upper bound can be obtained by considering a newsboy problem with understocking cost equal to $\text{Max}\{p - c_k - r_{z+1}, 0\}$ and overstocking cost equal to $c_k - s$. The critical ratio for this

newsboy problem is equal to $\text{Max}\{\frac{p-c_k-r_{z+1}}{p-r_{z+1}-s}, 0\}$. This leads to the following Theorem. The proof is analogous to that of Theorem 4.2 .

Theorem 4.3: If the procurement cost can be described by the step-up form in (15) and $r_j = 0$ or $r_j = \text{Max}\{p-c_j, 0\}$, then any optimal stocking level y in any period $j \leq j_z$ is bounded in the following way:

$$H\left(j_k, n; \text{Max}\left\{\frac{p-c_k-r_{k+1}}{p-r_{k+1}-s}, 0\right\}\right) \leq y \leq H\left(j_k, n; \text{Max}\left\{\frac{p-c_k-r_{z+1}}{p-r_{z+1}-s}, 0\right\}\right), \quad (19)$$

where $k = \text{Min}\{i: j_i \geq j\}$, and $F(j, n; x)$ is as defined in (17a-b).

5. Numerical Computations

In this section we present computational results for the dynamic programming model developed in Section 3. We consider the case where demands arriving when the broker is out of stock are satisfied without backlogging cost if it is profitable to do so, by setting $r_j = \max(p-c_{j+1}, 0)$. Note that this case could arise in situations such as the travel agent's where all demand is satisfied at the end of the horizon. Alternatively, it could also arise in situations where brokers procure many items from a single manufacturer so that deliveries occur on a frequent and regular schedule. In such situations, there would be little or no incremental cost associated with ordering an additional item.

For this case, we examine two classes of cost functions: step-up and linearly increasing. In addition to the fact that these cost functions are simple and commonly used in practice, the comparison of their effects provides a further understanding of the tradeoffs involved in our model.

We first examine how the optimal ordering policy and the associated expected profit differ between the two classes, as well as how they are affected by the parameters of the cost function. In both cases, the length of the selling season is equal to $N=50$, the retail price $p=25$, the salvage value $s=1$, and the prior distribution parameters for p are $\alpha=3$ and $\beta=5$ which corresponds to a prior estimate for the probability of an arrival equal to $3/8$.

Consider the case of a step-up cost function with a one-time increase, as defined in Figure 1. The percentage discount off the full price offered before time t_1 is equal to $100(c_f - c_d) / c_f$. We examine how the total expected profit of an optimal ordering policy depends on the discount and its duration j_1 , for a fixed value $c_f=20$. Figure 3 presents the optimal expected profit as a function of the discount for values of j_1 equal to 1 and 25, that is 2% and 50% of the selling season. Observe in Figure 3 that, as expected, the optimal profit is increasing both in d and j_1 . In addition, it is interesting to observe that the gap between the profit curves is first increasing then decreasing in the magnitude of the discount. Thus, the broker is most sensitive to the length of time that the discount is offered for discounts of intermediate magnitude. This can be explained as follows: For a very small discount, the broker has little to gain by altering his ordering

behavior in response to it regardless of the length of time for which it is available. On the other hand, for a sufficiently large discount, the broker can afford to place a large speculative order even with very poor information about demand. When the value of the discount is very large, it is so inexpensive to over-order that additional time, i.e. information, is not terribly valuable. For a discount between these two extremes, the value is significant enough to alter the broker's ordering behavior, but not so significant that the cost of misjudging demand is trivialized. Thus, the broker tends to value information, i.e. time, the most for a discount of intermediate magnitude.

We next consider the case where the wholesale cost increases linearly with j . In order to make comparisons between the step-up and linear cost cases, for every instance of the step-up case that we solve, we set the parameters of the linear cost in such a way that the average wholesale cost over the entire selling season $1, \dots, N$ is the same for both cases. Specifically, for a step-up cost function with parameters c_d , c_f and t_1 , we consider a corresponding linear cost function of the form $c_j = c_d + \theta(j-1)$, $j=1, \dots, N$, where $\theta = 2(c_f - c_d)(N-j_1)/N^2$ is the value of the slope that ensures equality between the average wholesale costs. Figure 4 presents, for $j_1 = 25$ and various values of d , the expected profit under a step-up cost function with discount equal to d and discount duration j_1 , and under the corresponding linear cost function. It can be seen in Figure 4 that the difference in profit between the two cases for this set of parameters is small. In general, our numerical computations show that the linear cost function results in higher expected profits only for extremely small values of the discount period j_1 . This can be explained in terms of the diminishing incremental information associated with the time that the broker has to observe demand. As time passes, the broker learns less and less from each additional unit of time for which demand is observed. Thus, unless the step-up discount period is very short, the broker would prefer to have access to the cheapest possible price for a short amount of time than to have access to a continuously increasing price for a longer period of time. Only when the step-up discount period is very short does the broker prefer the opportunity to delay ordering beyond the end of the period and pay the continuously increasing price.

It is also of interest to consider how these two cost functions affect the broker's order pattern. For this purpose we examine the orders prescribed by the optimal policy for a single realization of the demand process, under a step-up and its associated linear cost function. As in the previous example, assume that $N=50$, $c_f=10$ and $s=1$. In addition, the discount level for the step-up function is $d=36\%$, while the length of the discount period is $j_1=20$. In Figure 5, the optimal stocking levels associated with a single sample path are presented for the two cost functions. The instances of the demand occurrences are also included as small marks below the horizontal axis. Although the graphs in Figure 5 correspond to a single sample-path of the demand arrival process, they are representative of the behavior of the order policy. It can be observed in these graphs that the optimal policy places higher orders during the early season under the linear than under the step-up cost function. This is in agreement with the previous observations regarding the profit difference between the step-up and linear cases. It also implies that a linearly increasing cost function provides the

manufacturer with more information about the demand process sooner in the selling season than a single step-up cost function.

In the last computational experiment we examine the benefit of the adaptive ordering policy (A) developed in Section 2, with respect to two simpler policies: A *no-recourse* policy (NR) in which there a single order is placed at the beginning of the horizon, and a *single-recourse* policy (SR) in which there is a single opportunity for recourse at the end of the horizon after an initial order is placed. The optimal single-recourse policy as described above can be determined as the solution to a newsboy problem with over-order cost equal to c_1-s , and under-order cost equal to $\min(p-c_1, c_N-c_1)$. Figure 6 presents the relative expected profits associated with policies NR, SR and A for a step-up cost function with varying discount d and discount duration equal to 50% of the selling season. Obviously the broker's profits are highest under the adaptive policy, and are lowest under the no-recourse policy. It is interesting to observe that the gap between single-recourse and no recourse is decreasing in the magnitude of the discount. This is because, for large discounts, the broker can compensate for a lack of information or opportunity for recourse by over-ordering at the beginning of the horizon. It is also of interest to observe that the gap between the adaptive profits and the single recourse profits is greatest for discounts of intermediate magnitudes. This is due to the fact that for small discount values the cost function is almost flat therefore the adaptive policy does not offer significant benefits. On the other hand, for large discounts, the single-recourse policy can afford to order a lot at the beginning, suffering little by having excess inventory at the end of the horizon.

6. Conclusions and Extensions

In this paper, we have examined an important trade-off found in many short-life cycle products in which a firm (i.e. a broker) would like to observe as much demand as possible before making commitments to providers. However, this desire must be balanced against the incentives for early orders, provided by upstream providers who are eager to schedule their own operations. To gain insight into this trade-off, we have modeled demand as a Bernoulli process with unknown parameter and a finite horizon, and assumed that the per-unit cost of procurement is non-decreasing in time.

For the most general version of this problem, we have demonstrated that an optimal stocking policy can be characterized by an order-up-to threshold value that depends on the amount of elapsed time and the number of demands that have already occurred. Moreover, we have shown that, for a given number of demands, the level of the threshold is non-increasing (non-decreasing) in the amount of time that has passed (remains).

We have also analyzed the special case of a non-decreasing cost function, in which the cost faced by the broker increases only at a single period. Using the analysis for this special case, which is common in

practice, we derived simple bounds on the optimal threshold values for the optimal stocking policy under a cost function with multiple periods of increases.

In computational experiments, we demonstrated that for a single step-up cost function, the broker's optimal profits are most sensitive to the time duration of the discount when the magnitude of the discount is of intermediate value. That is, a change in the time duration of the discount affects the broker's profits more for medium discounts than for either very small or very large ones. This observation has interesting implications for the negotiation of contracts between brokers and providers. As the value of the discount is increased, brokers should first become more aggressive about asking for longer time durations, but after a certain threshold discount value is reached, they should become less aggressive in their demands for longer time durations.

In comparisons between comparable single step-up and linearly increasing cost functions, we demonstrated that, for all but the shortest discount periods, the broker's profits are higher with the step-up wholesale cost functions. This observation is consistent with the notion that the broker learns less and less from each additional observation of demand. However, we also observed that the broker tends to order more units earlier with the linear cost function. Thus, by using more frequent, smaller price increases, the providers can induce more information about end demand from the brokers early in the horizon.

Clearly, the two most restrictive of our assumptions are the absence of a fixed ordering cost and that the cost function is deterministic and non-decreasing. The absence of fixed ordering costs is quite reasonable in a setting, such as the travel agent's, in which all goods and services are exchanged at the end of the horizon. In such settings, the broker does not necessarily hold physical inventory. As a result, each "order" does not require a physical transaction, it only increases the level of commitment that the broker has made to her provider. Alternatively, in settings where brokers procure many items from a single manufacturer, there may be little or no incremental cost associated with adding another item to a regularly scheduled delivery. However, there are many other settings in which there are significant fixed costs associated with the placement of orders. Incorporating these these fixed ordering costs would certainly be a valuable direction for future research.

Our results are also based on the assumption that the broker's cost of procurement is non-decreasing and known in advance. Non-decreasing costs are reasonable in many settings where the providers are not perfectly competitive and want to avoid creating expectations of future price decreases. Note that this characterization is accurate in the airline industry where it is very rare for fares to decrease as the date of travel approaches. The assumption that procurement cost increases are known in advance may also be realistic in settings where the increases are based on premium shipping costs (eg. a more expensive mode of transportation may be required as the end of the horizon approaches.) Nevertheless, there are instances in which (from the broker's perspective), cost increases occur at random, or as a function of the observed demand, or even as a function of the broker's placed orders. The last case gives rise to game theoretic models that capture the provider's pricing policy as a response to the broker's orders. It is not

likely that our results are immediately extended in such cases. On the other hand, it is likely that the optimality of an order-up-to policy and most of our other structural results would continue to hold for stochastic cost increases that are either independent of demand or are stochastically increasing in demand. Certainly, one would expect that uncertainty about when an increase will occur would tend to smooth out the broker's ordering pattern. For example, if the times at which price increases would occur were uncertain, we would not see the same clumping of orders at a single point in time as we do when they are known in advance. On the other hand, if the timing of increases were known and only the magnitudes of the increases were subject to uncertainty, we would most likely continue to observe clustering of orders immediately before time periods in which increases occur. It is hoped that this paper can serve as a building block for future models that formally consider either stochastic procurement cost increases or fixed ordering costs.

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Appendix

Proposition 3.1: The optimal profit $v_{j+1}(n,x)$ is non-decreasing in the stock level x .

Proof: We will show that $d_{j+1}(n,x) \geq 0$ by backward induction on j . For $j=N$ it follows from (3b), $d_{N+1}(n,x) = s \geq 0$. Assume that, for some $j \leq N-1$, $d_{j+2}(n,x) \geq 0$ for all n,x . We now consider two cases for $v_{j+1}(n,x)$.

Case 1: $\text{Max}_{y \geq x} \{w_{j+1}(n,y)\} = w_{j+1}(n,x)$, thus, $v_{j+1}(n,x) = c_{j+1}x + w_{j+1}(n,x)$.

In this case, since $v_{j+1}(n,x+1) \geq c_{j+1}(x+1) + w_{j+1}(n,x+1)$, it follows that $d_{j+1}(n,x) \geq c_{j+1} + \delta_{j+1}(n,x)$. In addition, from (1), $r_{j+1} \leq p$, thus, from (8a) and (8b), $\delta_{j+1}(n,x) \geq -c_{j+1}$, for all $x \geq 0$. Hence, $d_{j+1}(n,x) \geq 0$.

Case 2: $\text{Max}_{y \geq x} \{w_{j+1}(n,y)\} = w_{j+1}(n,x')$ for some $x' > x$.

Then, it is also true that $\text{Max}_{y \geq x+1} w_{j+1}(n,y) = w_{j+1}(n,x')$, therefore, $d_{j+1}(n,x) = c_{j+1} > 0$.

This completes the induction proof. ◆

Proposition 3.2: The differences $\delta_{j+1}(n,y)$, $d_{j+1}(n,y)$ are non-decreasing in n , for all $j \leq N$ and all y .

Proof: The proof will be by backward induction on j . The claim is easily seen to be true for $j=N-1$. Assume that for some $j \leq N-1$, $\delta_{j+2}(n,y)$, $d_{j+2}(n,y)$ are non-decreasing in n . We have from (8a) that, for $y > 0$,

$$\begin{aligned} \delta_{j+1}(n+1,y) &= -c_{j+1} + \frac{\alpha+n+1}{\alpha+\beta+j} d_{j+2}(n+2,y-1) + \left(1 - \frac{\alpha+n+1}{\alpha+\beta+j}\right) d_{j+2}(n+1,y) \\ &= -c_{j+1} + \frac{\alpha+n}{\alpha+\beta+j} d_{j+2}(n+2,y-1) + \left(1 - \frac{\alpha+n}{\alpha+\beta+j}\right) d_{j+2}(n+1,y) \\ &\quad + \frac{1}{\alpha+\beta+j} (d_{j+2}(n+2,y-1) - d_{j+2}(n+1,y)). \end{aligned}$$

From the induction hypothesis and the concavity of $v_{j+2}(n,y)$ in y , we have that

$$d_{j+2}(n+2,y-1) \geq d_{j+2}(n+1,y-1) \geq d_{j+2}(n+1,y).$$

Therefore,

$$\delta_{j+1}(n+1, y) \geq -c_{j+1} + \frac{\alpha+n}{\alpha+\beta+j} d_{j+2}(n+2, y-1) + \left(1 - \frac{\alpha+n}{\alpha+\beta+j}\right) d_{j+2}(n+1, y),$$

and using the induction hypothesis again together with (8a), $\delta_{j+1}(n+1, y) \geq \delta_{j+1}(n, y)$.

Similarly, for $y=0$,

$$\begin{aligned} \delta_{j+1}(n+1, 0) &= -c_{j+1} + \frac{\alpha+n+1}{\alpha+\beta+j} (p-r_{j+1}) + \left(1 - \frac{\alpha+n+1}{\alpha+\beta+j}\right) d_{j+2}(n+1, 0) \\ &= -c_{j+1} + \frac{\alpha+n}{\alpha+\beta+j} (p-r_{j+1}) + \left(1 - \frac{\alpha+n}{\alpha+\beta+j}\right) d_{j+2}(n+1, 0) \\ &\quad + \frac{1}{\alpha+\beta+j} (p-r_{j+1} - d_{j+2}(n+1, 0)). \end{aligned}$$

From Theorem 3.1, part (a), $d_{j+2}(n+1, 0) \leq p-r_{j+1}$, while, by the induction hypothesis, $d_{j+2}(n+1, 0) \geq d_{j+2}(n, 0)$.

Thus, $\delta_j(n+1, 0) \geq \delta_j(n, 0)$, and the proof is complete. \blacklozenge

Proposition 3.3: $y_{j+1}''(n)$ is non-increasing in j for all n .

Proof: We will show that $y_{j+1}''(n) \geq y_{j+2}''(n)$ for all j, n . The proof is by contradiction.

We first consider the case when $y_{j+1}''(n) > 0$. From (8a) and (10),

$$\begin{aligned} d_{j+1}(n, y_{j+1}''(n)) &= c_{j+1} + \delta_{j+1}(n, y_{j+1}''(n)) \\ &= \frac{\alpha+n}{\alpha+\beta+j} d_{j+2}(n+1, y_{j+1}''(n)-1) + \left(1 - \frac{\alpha+n}{\alpha+\beta+j}\right) d_{j+2}(n, y_{j+1}''(n)). \end{aligned} \quad (11)$$

Assume $y_{j+1}''(n) < y_{j+2}''(n)$. From Corollary 3.1 it is also true that $y_{j+2}''(n) \leq y_{j+2}''(n+1)$. Therefore, by the definition of $y_{j+2}''(n)$, it follows that $\delta_{j+2}(n, y_{j+1}''(n)) \geq 0$ and $\delta_{j+2}(n, y_{j+1}''(n)-1) \geq 0$. Thus, from (10), we have that $d_{j+2}(n, y_{j+1}''(n)) = d_{j+2}(n, y_{j+1}''(n)-1) = c_{j+2}$. We now obtain from (11) $c_{j+1} + \delta_{j+1}(n, y_{j+1}''(n)) = c_{j+2}$, which is a contradiction since, from the definition of $y_{j+1}''(n)$, it follows that $\delta_{j+1}(n, y_{j+1}''(n)) < 0$, and we have assumed $c_{j+1} \leq c_{j+2}$.

The case $y_{j+1}''(n) = 0$ can be shown similarly, using (8b) and (10). \blacklozenge

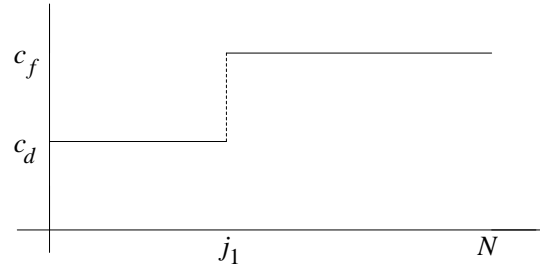


Figure 1: Retailer's Wholesale Cost: Single Step-up Cost Function

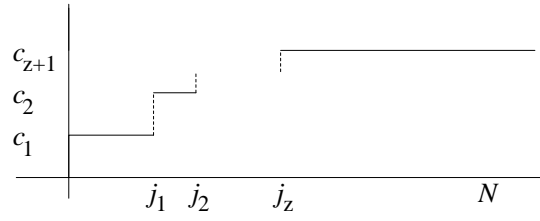


Figure 2: Retailer's Wholesale Cost: General Step-up Cost Function

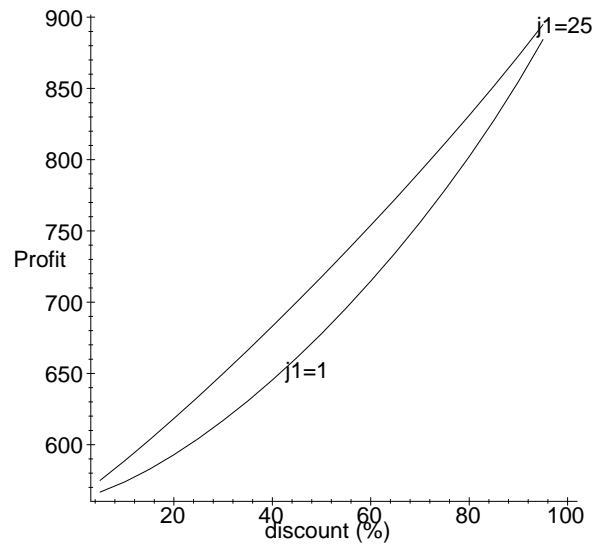


Figure 3: Optimal Profit for Stepup Purchase Cost Function

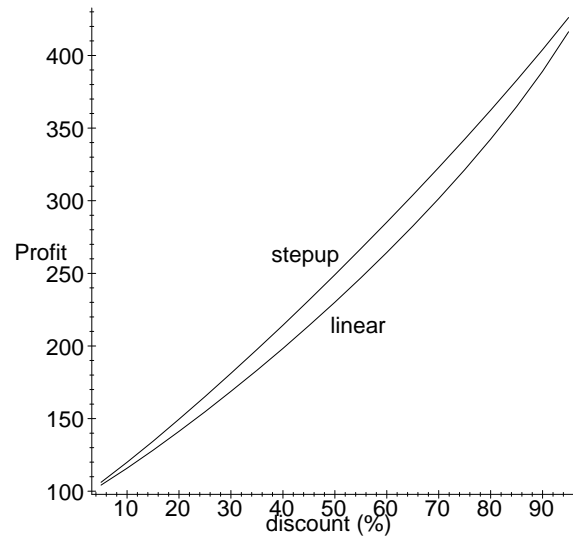


Figure 4: Profit Comparison Under Stepup and Linear Cost Functions

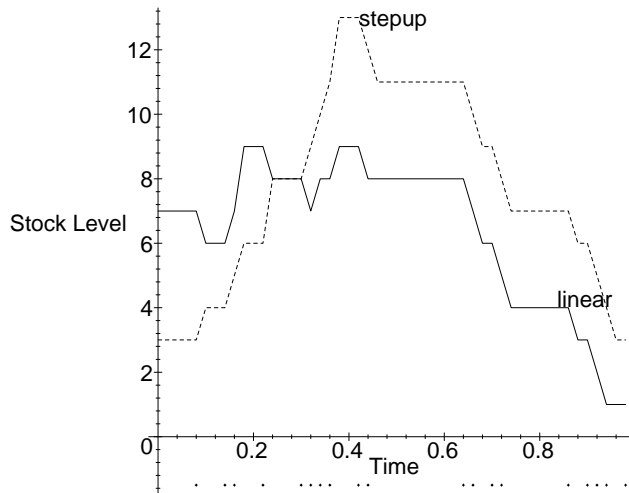


Figure 5: Comparison of Ordering Policies on a Sample Path

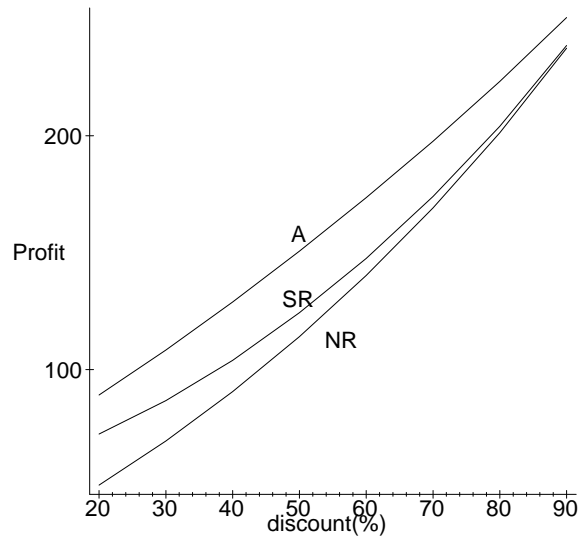


Figure 6: Comparison of Adaptive and Non Adaptive Ordering Policies