**Technical Memorandum Number 726**

# Dynamic Policies of Admission to a Two-Class System Based on Customer Offers

**by**

**E. Lerzan Örmeci**
**Apostolos N. Burnetas**
**Hamilton Emmons**

**April 2000**

**Department of Operations**
**Weatherhead School of Management**
**Case Western Reserve University**
**10900 Euclid Avenue**
**Cleveland, Ohio 44106-7235**

# Dynamic Policies of Admission to a Two–Class System Based on Customer Offers

E. Lerzan Örmeci, Apostolos Burnetas and Hamilton Emmons

April 17, 2000

**Abstract**

We consider the problem of dynamic admission control in a Markovian loss queueing system with two classes of jobs with different service rates and random revenues. We establish the existence of an optimal monotone policy. We also show that under certain conditions there exist preferred jobs from either class.

## 1 Introduction

In this paper, we consider the problem of controlling the workload in a production or service system, via dynamic policies of accepting incoming jobs. Problems of this type arise in situations where arriving jobs differ in service requirements and the profit they generate. The profit differentiation is due to the fact that with each job is associated a different compensation, that generally depends on the resources required to complete it as well as on the amount that the customer is prepared to pay for the particular job. For example, customers may offer different prices for jobs of similar types, because the importance of each job in the customer's own operation may be different. On the other hand, the system owner may charge a fixed entrance fee for jobs of a specific type, but the actual cost of serving each job may vary due to raw materials, labor, etc. In such situations, the system owner can benefit from basing his/her acceptance decisions on the current utilization of the system resources and the profit accompanying each incoming job. This calls for dynamic rather than static admission control rules.

We study the issues mentioned above in the context of a system with $c$ identical parallel servers, no waiting room and two types of jobs (referred to in queuing terminology as a two-class loss system). Class-$i$ jobs arrive at the system according to a Poisson process with rate $\lambda_i$ and demand an exponential service time with mean $1/\mu_i$, where $\mu_1 \leq \mu_2$. The reward associated with admitting a class-$i$ job is a random variable $\rho_i$. The actual value of $\rho_i$ is known before the admission decision is made, and, if the job is admitted to the system, the reward is collected at the beginning of service. We assume that $\rho_i$ follows probability distribution $F_i$ with finite mean, and the rewards of successive jobs are independent.

The motivation behind the assumption of no waiting buffer is to isolate the effect of admission control on performance. Indeed, in systems where job waiting is allowed, there is the additional issue of scheduling waiting jobs, and the two effects are difficult to separate. From the practical point of view, this assumption is consistent with recent trends in manufacturing to utilize admission restrictions rather than scheduling and routing in order to control work in process inventories (see for example Ku & Jordan (1997)), an objective shared by just in time manufacturing.

We are interested in dynamic admittance policies that maximize the total expected discounted reward with continuous discount rate $\beta$ over an infinite horizon as well as the long-run average net profit. We establish several structural results of optimal policies under these criteria. We first observe that an optimal policy admits an incoming job only if the associated reward exceeds a minimal threshold, which depends on the job class and the current system load. Then, we prove that the acceptance threshold for jobs of class $j$ is increasing in the number of class-$k$ jobs that are currently in the system, with $k \neq j$. Therefore, everything else being equal, it becomes more difficult to accept an incoming job, when more jobs of the other class are currently under service. Finally, we show that under certain conditions, there are jobs from either class that offer sufficiently high rewards so that they are never denied service unless all servers are occupied. Such jobs are hereafter referred to as "preferred".

There has been an increasing interest in multiclass loss networks recently, due to growth in telecommunications systems. Admission control is one of the main research areas on loss networks; see Chapter 4 of Ross (1995) for a comprehensive review. Most of the research in the area has concentrated on certain types of policies (e.g., coordinate convex policies, trunk reservation policies) rather than analyzing the optimal policy directly. We note the following exceptions. Keilson (1970) and Lippman & Ross (1971) described the optimal admission rule for a system with one server and no waiting room which receives offers from customers according to a joint service time and reward probability distribution. Miller (1971) considers a system with $c$ parallel identical servers, no waiting room and $k$ job classes. All jobs demand an exponential service with the same rate, whereas they offer fixed rewards depending on their class. Ku & Jordan (1997) consider two stations in tandem each with no waiting room and parallel servers. Carrizosa, Conde & Munoz-Marquez (1998) present an optimal static control policy for acceptance/rejection of $k$ classes in an $M/G/c/c$ queue, where each class has a different service requirement and a different reward. Örmeci, Burnetas & van der Wal (1999) consider the same system as in this paper, with the difference that rewards are fixed for each class. Deterministic, class-dependent rewards are assumed in all these studies, except for Lippman & Ross (1971). However, random rewards have been considered in queueing systems; e.g., Ghoneim & Stidham (1985) analyze the optimal admission policies for a system with two queues in series and two classes of jobs that bring random rewards and require different services.

If we assume that the rewards are determined and collected at the end of service, then the system is equivalent to one in which jobs of class $i$ offer a fixed reward $r_i = E[\rho_i]$. This system is analyzed in Örmeci et al. (1999), where, in addition to the structure of the optimal admission policy, there is also a discussion on the existence of preferred job classes. It is shown there that, under certain conditions, a preferred class can be identified by a scheme analogous to the well-known $c\mu$ rule in scheduling. The present paper generalizes Örmeci et al. (1999) by assuming a probability distribution for the reward of each class, and allowing the admission policy to depend on the actual reward offered by an incoming job. Under these circumstances, it is not expected that all jobs of a class will be preferred, because jobs of the same class are further distinguished upon arrival by their reward. As a consequence, we can only identify conditions under which some jobs of any class are preferred.

The admission of jobs into the system can also be controlled via pricing. Instead of rejecting the jobs, the system owner can announce a price which may depend on the state of the system and/or the job class, for which the incoming jobs are accepted for service. This kind of control has been considered in the context of social optimization for various systems; see e.g., Naor (1969), Lippman & Stidham (1977) and Xu & Shantikumar (1993). For loss systems, Miller & Buckman (1987) consider a static transfer pricing problem for a single class $M/M/c/c$ system.

This paper is organized as follows: In the next section, we present the corresponding Markov decision process model of the system described above. The third section proves the existence of an optimal monotone policy. The fourth section presents the conditions under which preferred jobs exist and how to determine them. Finally, we discuss possible extensions in the last section.

## 2 Markov Decision Model

In this section, we model the admission control problem as a continuous time Markov decision process with expected total discounted reward criterion. We then use standard methodology to develop an equivalent model in discrete time with undiscounted expected reward objective.

The state of the system must include the number of jobs of each class being served. The events that cause the state to change are service completions and admissions of arriving jobs. Because the decision to admit an arriving job is allowed to depend not only on the number of jobs present but also on the type and reward of the new job, this information must also be incorporated in the state description. We therefore define the state at arrival instants as $(x; j, \rho_j) = (x_1, x_2; j, \rho_j)$ when there is a new job arrival of class $j$ with reward equal to $\rho_j$, and there are $x_i$ jobs of class $i$ currently in the system, $i = 1, 2$. At all other times the state information is described by $x = (x_1, x_2)$. Note that in both definitions $x$ is restricted to the set $\mathcal{S} = \{x \in \mathcal{Z}^2 : x \geq 0, x_1 + x_2 \leq c\}$, where $\mathcal{Z}$ is the set of integers.

Admission decisions are made only at instants of arrivals; the corresponding action sets are $A(x; j, \rho_j) = \{0, 1\}$ with 1 denoting the decision to admit the arriving class-$j$ job and 0 to reject it. The reward structure is described solely by the immediate rewards earned when new jobs are admitted, i.e., $R((x; j, \rho_j), 1) = \rho_j$ and $R((x; j, \rho_j), 0) = 0$.

The transition mechanism is defined as follows. If in state $(x; j, \rho_j)$ the incoming job is admitted, the state is changed instantaneously to $(x + e_j)$, where $e_j$ is the two-dimensional unit vector with the $j^{\text{th}}$ component equal to 1. If the job is not admitted, the state remains in the same state before the arrival, i.e., in $x$. From state $x$, where no decision is made, transitions are triggered either by job arrivals or by service completions, which occur with exponential rate. Specifically, a transition corresponding to arrival of a job of class $j$ occurs with exponential rate $\lambda_j$, and in this case the new state is further determined by the associated reward $\rho_j$ generated from the density $f_j$. Similary, a service completion of a job of class $j$ occurs with exponential rate $x_j\mu_j$. Therefore, the exponential rate of transition out of state $x$ equals $q(x) = \lambda_1 + \lambda_2 + x_1\mu_1 + x_2\mu_2$.

The continuous time model is completely specified by the above description, together with the continuous discount rate $\beta$. The following transformations will simplify the analysis. First, the model is equivalent to one in which for every state $x$ there is an additional exponential transition rate equal to $\beta$ to a new state $s$ (stopped state). Therefore, the equivalent transition rate out of state $x$ becomes $\tilde{q}(x) = q(x) + \beta$. The objective in the new system is the total expected undiscounted reward, until the first passage to state $s$. (For the equivalence of the discounted process and the undiscounted first passage process see e.g., Walrand (1988)).

The model can be further transformed into one in essentially discrete time, via the well known method of uniformization (see, e.g., Lippman (1975$a$)). Since $\mu_1 \leq \mu_2$, it follows that $\tilde{q}(x) \leq M \equiv \lambda_1 + \lambda_2 + c\mu_2 + \beta$ for all $x$. We define a transformed model, in which the exponential transition rate out of any state $x$ is equal to $M$. When a transition occurs from state $x$ in the new model, it is a class-$j$ arrival with probability $\lambda_j/M$, a class-$i$ service completion with probability $x_i\mu_i/M$, a transition to the stopped state $s$ with probability $\beta/M$, or a "transition" back to state $x$ with probability $\left(1 - (\lambda_1 + \lambda_2 + x_1\mu_1 + x_2\mu_2 + \beta)\right)/M = (c\mu_2 - x_1\mu_1 - x_2\mu_2)/M$. We refer to these last transitions as "fictitious" service completions. We further assume without loss of generality that $M = 1$, which is equivalent to a time rescaling so that a unit of time is equal to the expected time between transitions.

We have so far established that, in order to analyze the optimal admission policy, we can consider an equivalent system with undiscounted expected reward criterion, in which the times between transitions are exponentially distributed with rate 1. Since no discounting is involved, it is finally sufficient to consider only the imbedded process of transitions instances, in which the time between transitions is exactly equal to 1, and the transformation to a discrete time model is complete.

We now develop the optimality equations for the transformed system, in finite horizon. Let $u_n(x)$ and $v_n(x; j, \rho_j)$ be the maximal expected reward, starting in state $x$ and $(x; j, \rho_j)$, respectively, until state $s$ is reached, or $n$ transitions occur, whichever happens first. Computing $v_n(x; j, \rho_j)$ requires comparison of two actions: accepting the incoming class-$j$ job which implies moving to state $x + e_j$ with a reward of $\rho_j$, and rejecting it so that the system remains in state $x$ with no reward. Define $a_n(x; j, \rho_j)$ to be the optimal action in state $(x; j, \rho_j)$ with $n$ remaining transitions.

The optimality equations of this model are as follows. For $x_1 + x_2 < c$:

$$
\begin{aligned}
v_n(x; j, \rho_j) &= \max\{\rho_j + u_n(x + e_j), u_n(x)\} \qquad (1) \\
u_{n+1}(x) &= \lambda_1 E[v_n(x; 1, \rho_1)] + \lambda_2 E[v_n(x; 2, \rho_2)] + \\
&\quad x_1\mu_1 u_n(x - e_1) + x_2\mu_2 u_n(x - e_2) + \\
&\quad (c\mu_2 - x_1\mu_1 - x_2\mu_2) u_n(x),
\end{aligned}
$$

where we set $u_n(-1, x_2) = u_n(0, x_2), u_n(x_1, -1) = u_n(x_1, 0)$, and $E[h(x; j, \rho_j)]$ denotes expectation with respect to the probability distribution $F_j$. For $x_1 + x_2 = c$, no jobs can be accepted so that $a_n(x; j, \rho_j) = 0$, and thus $v_n(x; j, \rho_j) = u_n(x)$.

The results on the structure of the optimal admission policy will be proved for all finite horizons $n$ (typically with induction arguments). From the standard theory of discounted Markov decision processes it then follows that these results can be extended to the case of infinite horizon, since the following limits exist for $\beta > 0$:

$$
\begin{aligned}
v(x; j, \rho_j) &= \lim_{n\to\infty} v_n(x; j, \rho_j) \\
a(x; j, \rho_j) &= \lim_{n\to\infty} a_n(x; j, \rho_j) \\
u(x) &= \lim_{n\to\infty} u_n(x).
\end{aligned}
$$

Finally, the structural results derived for the discounted problem carry over to the case of maximizing the expected long-run average return. This can be established by letting the discount rate $\beta$ approach zero, and using the fact that the model is unichain since state (0,0) is reachable from all states. Then, it is straightforward to verify that the conditions introduced by Lippman (1975*b*) are satisfied. For details on the existence of the infinite horizon limits, see Örmeci (1998).

We conclude this section introducing some notation that will be useful in the subsequent analysis of the optimality equations. Let

$$D_n(ij)(x) = u_n(x+e_i) - u_n(x+e_j), \quad i = 0,1,2, \; j = 1,2,$$

where we set $e_0 = (0,0)$. The quantity $D_n(ij)(x)$ is equal to the relative benefit of starting in state $x+e_i$ vs. $x+e_j$, with a horizon of $n$ transitions. It is now easy to see from equation (1) that

$$a_n(x; j, \rho_j) = 1 \iff \rho_j \geq D_n(0j)(x),$$

with the assumption that we choose to serve a job, if both rejecting and accepting it is optimal. Hence, it is optimal to accept the incoming job in state $(x; j, \rho_j)$ if and only if the reward it brings exceeds a threshold equal to $D_n(0j)(x) = u_n(x) - u_n(x+e_j)$, which represents the loss in future rewards because of the increased load if the job is accepted, or, equivalently, the expected burden that an additional class-$j$ job brings to the system in state $x$ when there are $n$ remaining transitions. Similarly, the difference $D_n(21)(x)$ represents the expected additional burden, when in state $x+e_2$, of changing a class-2 job that is already in the system to a class-1 job. In the following sections, we may omit $n$ and $x$ from the expression $D_n(ij)(x)$ when there is no danger of confusion.

## 3 Structure of the optimal policy

We have observed that $D_n(0j)(x)$ determines a threshold on the offer by a class-$j$ job to be accepted when in state $x$. In this section, we prove that this minimal acceptable offer, $D_n(0j)(x)$, is increasing in the number of class-$k$ jobs in the system, for $k \neq j$. Therefore, it is more difficult for a class-$j$ job to be admitted when there are more class-$k$ jobs in the system.

**Lemma 1** *For all $x$ such that $x_1 + x_2 + 2 \leq c$,*

$$D_n(01)(x) - D_n(01)(x+e_2) = D_n(02)(x) - D_n(02)(x+e_1) \leq 0 \quad \forall n \geq 1, \qquad (2)$$

*provided that initial conditions $u_0(x)$ also satisfy (2) for $n = 0$ and are otherwise arbitrary.*

**Proof.** From the definition of $D_n(ij)(x)$ it follows that

$$D_n(01)(x) - D_n(01)(x+e_2) \quad = \quad D_n(02)(x) - D_n(02)(x+e_1)$$

$$= \quad u_n(x) - u_n(x+e_2) - u_n(x+e_1) + u_n(x+e_1+e_2).$$

Assume that $u_0$ satisfies (2). Note that many functions satisfy this inequality including the usual choice $u_0(x) = 0$.

We prove the lemma by induction on the number of remaining transitions. Assume that the statement is true for $n$. We first show that $v_n(x; j, \rho_j)$ also satisfies this monotonicity. Let

$$\delta(x; j) = v_n(x; j, \rho_j) - v_n(x+e_1; j, \rho_j) - v_n(x+e_2; j, \rho_j) + v_n(x+e_1+e_2; j, \rho_j)$$

We first consider when there is a class-1 arrival; there are four possible cases due to the actions $a_n(x; 1, \rho_1)$ and $a_n(x+e_1+e_2; 1, \rho_1)$:

**Case I:** $a_n(x; 1, \rho_1) = a_n(x+e_1+e_2; 1, \rho_1) = 0$. In this case,

$$\delta(x; 1) \leq u_n(x) - u_n(x+e_1) - u_n(x+e_2) + u_n(x+e_1+e_2) \leq 0,$$

where the first inequality follows from the case assumptions and the optimality equation for $v_n$, and the second one from the induction hypothesis.

**Case II:** $a_n(x; 1, \rho_1) = 1$ and $a_n(x+e_1+e_2; 1, \rho_1) = 0$. In this case,

$$\delta(x; 1) \leq u_n(x+e_1) + \rho_1 - u_n(x+e_1) - u_n(x+e_1+e_2) - \rho_1 + u_n(x+e_1+e_2) = 0,$$

where the inequality follows from the case assumptions and the optimality equations for $v_n$.

**Case III:** $a_n(x; 1, \rho_1) = 0$ and $a_n(x+e_1+e_2; 1, \rho_1) = 1$. In this case,

$$\begin{aligned} \delta(x; 1) &\leq u_n(x) - u_n(x+2e_1) - \rho_1 - u_n(x+e_2) + u_n(x+2e_1+e_2) + \rho_1 \\ &= u_n(x) - u_n(x+e_2) - u_n(x+e_1) + u_n(x+e_1+e_2) \\ &\quad + u_n(x+e_1) - u_n(x+e_1+e_2) - u_n(x+2e_1) + u_n(x+2e_1+e_2) \leq 0, \end{aligned}$$

where the first inequality is true due to the case assumptions and optimality equations and the second one due to the induction hypothesis.

**Case IV:** $a_n(x; 1, \rho_1) = 1$ and $a_n(x+e_1+e_2; 1, \rho_1) = 1$. In this case,

$$\delta(x;1) \le u_n(x+e_1) - u_n(x+2e_1) - u_n(x+e_1+e_2) + u_n(x+2e_1+e_2) \le 0,$$

where the first inequality is true due to the case assumptions and the optimality equations, and the second one due to the induction hypothesis.

Hence, $v_n(x;1,\rho_1)$ satisfies inequality (2) for all $\rho_1$. We observe that:

$$E[\delta(x;j)] = E[v_n(x;j,\rho_j)] - E[v_n(x+e_2;j,\rho_j)] - E[v_n(x+e_1;j,\rho_j)] + E[v_n(x+e_1+e_2;j,\rho_j)].$$

Thus, $E[v_n(x;1,\rho_1)]$ also satisfies (2). $E[v_n(x;2,\rho_2)]$ can be similarly shown to satisfy (2). We next consider $u_{n+1}$:

$$\begin{aligned}
& u_{n+1}(x) - u_{n+1}(x+e_2) - u_{n+1}(x+e_1) + u_{n+1}(x+e_1+e_2) \\
= \; & \lambda_1 \; E[\delta(x;1)] + \lambda_2 \; E[\delta(x;2)] \\
& + x_1\mu_1 \; [u_n(x-e_1) - u_n(x-e_1+e_2) - u_n(x) + u_n(x+e_2)] \\
& + x_2\mu_2 \; [u_n(x-e_2) - u_n(x) - u_n(x+e_1-e_2) + u_n(x+e_1)] \\
& + \alpha \; [u_n(x) - u_n(x+e_2) - u_n(x+e_1) + u_n(x+e_1+e_2)] \\
\le \; & 0,
\end{aligned}$$

where $\alpha = c\mu_2 - (x_1+1)\mu_1 - (x_2+1)\mu_2$. The first two terms are less than or equal to 0 since $E[v_n]$ is shown to satisfy the inequality, and the remaining terms are also non-positive by the induction hypothesis. Thus, the value functions, $u_n$, satisfy inequality (2) for all $n$ whenever $u_0$ does. □

An immediate consequence of Lemma 1 is the following monotonicity property of an optimal admission policy:

**Theorem 1** *If it is optimal to reject a class-$j$ job in state $(x;j,\rho_j)$, then it is optimal to reject it in all states $(x+le_k;j,\rho_j)$ with $l \ge 1$ and $k \ne j$.*

The same result can expressed via threholds on the number of class-$j$ customers in the system when admission of a class-$k$ customer is considered, with $j \ne k$:

**Corollary 1** *For each $\rho_i > 0$, there exist thresholds $\{l_n^{i,\rho_i}(0), ..., l_n^{i,\rho_i}(c-1)\}_{\{i=1,2\}}$ such that:*

$$a_n(x;1,\rho_1) = \begin{cases} 0 & : \; x_2 \ge l_n^{1,\rho_1}(x_1) \\ 1 & : \; \textit{otherwise} \end{cases}$$

$$a_n(x;2,\rho_2) = \begin{cases} 0 & : \quad x_1 \geq l_n^{2,\rho_2}(x_2) \\ 1 & : \quad otherwise \end{cases}$$

# 4 Existence of preferred jobs

Recall that preferred jobs are those that are always admitted to the system whenever there is at least one idle server. In this section, we show that under certain conditions there are preferred jobs from either of the classes. This is natural in the sense that whenever customers are willing to pay sufficiently high prices for the jobs they need, they should receive immediate service if there is an available server.

We first define

$$\bar{\rho}_j = \min\{x : P\{\rho_j \leq x\} = 1\},$$

where we set $\min \emptyset = \infty$. In other words, $\bar{\rho}_j$ is the supremum of the reward that can be received from a class-$j$ job. Note that $\bar{\rho}_j$ is well defined because $P\{\rho_j \leq x\}$ is right continuous in $x$, therefore, the set $\{x : P\{\rho_j \leq x\} = 1\}$ either is empty or it has a minimum.

The existence of preferred jobs will be established by analyzing the behavior of the differences $D_n(ij)(x)$ in more detail. Specifically, in order to prove that there are preferred jobs of class $j$, it is sufficient to show that $D_n(0j)(x) < \bar{\rho}_j$ for all $x \in \mathcal{S}$, since in this case class-$j$ jobs that offer high enough rewards arrive at the system with positive probability $P\{D_{max}(0j) \leq \rho_j < \bar{\rho}_j\} > 0$.

In the proofs of the subsequent results we will make use of sample path arguments and coupling. Specifically, we will use the assumption that $\mu_1 \leq \mu_2$ (from which it follows that class-1 jobs are "slow"), to couple the service times of class-1 and class-2 jobs. To couple service times of a certain class-1 job, say $d_1$, and a class-2 job, say $d_2$, we let $\xi$ be a uniformly distributed random variable in $(0,1)$, and we generate the service times of $d_1$ and $d_2$ using the same $\xi$, so job $d_2$ has a shorter service time than job $d_1$ with probability 1. In terms of discrete time, this translates to the following: Both jobs leave the system with probability $\mu_1$, and a class-2 job departs from the system with probability $\mu_2 - \mu_1$ leaving the coupled class-1 job in the system. Thus, coupling does not allow a coupled class-1 job to leave the system while the coupled class-2 job is still there.

Lemma 2 below establishes the nonnegativity of $D_n(0j)(x)$'s and $D_n(21)(x)$. The intuition behind this property is simple. Recall that $u_n(x)$ is equal to the expected discounted total reward of the system under the optimal policy when there are $n$ more transitions. Thus, $u_n(x)$ is affected by the future rewards, and since rewards are collected at the beginning of service, jobs that are already in the system do not contibute to $u_n(x)$. In other words, the jobs initially in the system bring only more burden by blocking acceptance of more jobs. Hence, it is always preferable to be in a state where there are fewer or faster jobs.

**Lemma 2** *For $j = 1, 2$, for all $x \in \mathcal{S}$ and $n \geq 0$:*
*(1) $D_n(0j)(x) \geq 0$.*
*(2) $D_n(21)(x) = -D_n(12)(x) \geq 0$.*

**Proof.** We prove the statements by a sample path analysis.

(1) Assume that system A is in state $x$ and system B in $x+e_j$ in period $n$. We let system B follow the optimal policy, $\pi$, and system A imitate all the decisions of system B. We couple the two systems via the service and interarrival times, i.e., except for the additional job in system B, all the departure and arrival times are the same in both systems. We note that system A can always imitate system B since it always has at least as many free servers as system B does. Then, all future rewards of system A and B are the same:

$$D_n(0j)(x) = u_n(x) - u_n(x+e_j) \geq u_n^{\pi}(x) - u_n(x+e_j) = 0.$$

where $u_n^{\pi}(x)$ is the expected discounted return of system A.

(2) Assume that system A starts in state $x+e_2$ and system B starts in $x+e_1$, where we now couple the additional class-2 job, say job $d_2$, in system A with the additional class-1 job, say job $d_1$, in system B, as well as all other service and interarrival times, so that, as discussed earlier, if $d_1$ leaves the system, $d_2$ also leaves. Then, we can let system B follow the optimal policy and system A imitate all the decisions of system B. Now, again, all future rewards of both systems are equal:

$$D_n(21)(x) = u_n(x+e_2) - u_n(x+e_1) \geq u_n^{\pi}(x+e_2) - u_n(x+e_1) = 0,$$

with $u^{\pi}(x+e_2)$ the expected discounted return of system A. □

From Lemma 2 it also follows that $u(0,0) \geq u(x)$ for all $x \in \mathcal{S}$. This allows us to prove our next result, that the infinite horizon value function $u(x)$ is bounded even though $\rho_j$

can be unbounded. Let

$$\tilde{\rho} = \sum_{j=1,2} \lambda_j E[\rho_j].$$

Note that $\tilde{\rho} < \infty$, due to the assumption that rewards have finite mean.

**Lemma 3** *If $u_0(x) < \tilde{\rho}/\beta$ for all $x \in \mathcal{S}$, then $u_n(x) < \tilde{\rho}/\beta$ for all $x \in \mathcal{S}$ and $n \geq 1$, and $u(x) < \tilde{\rho}/\beta$ for all $x \in \mathcal{S}$.*

**Proof.** The proof will be by induction. By the assumption of the lemma, $u_0(x) < \tilde{\rho}/\beta$ for all $x \in \mathcal{S}$. Assume that the inequality is true for some $n$. Then, for $n+1$, it follows from from (1) that

$$\begin{aligned}
u_{n+1}(x) &= \sum_{j=1,2} (\lambda_j E[\max\{\rho_j + u_n(x+e_j), u_n(x)\}] + x_j \mu_j u_n(x-e_j)) \\
&\quad + (c\mu_2 - x_1\mu_1 - x_2\mu_2) u_n(x) \\
&\leq \sum_{j=1,2} (\lambda_j E\left[\max\{\rho_j + u_n(0,0), u_n(0,0)\}\right] + x_j \mu_j u_n(0,0)) \\
&\quad + (c\mu_2 - x_1\mu_1 - x_2\mu_2) u_n(0,0) \\
&= \sum_{j=1,2} (\lambda_j E[\rho_j + u_n(0,0)]) + c\mu_2 u_n(0,0) \\
&= \tilde{\rho} + (1-\beta) u_n(0,0) \leq \tilde{\rho} + (1-\beta)\frac{\tilde{\rho}}{\beta} = \frac{\tilde{\rho}}{\beta},
\end{aligned}$$

where the first inequality follows from Lemma 2, the second equality from the non-negativity of rewards $\rho_j$, the last equality from normalization, and the last inequality is due to the induction hypothesis.

Since $u(x) = \lim_{n\to\infty} u_n(x)$, $u(x)$ also satisfies the inequality. □

Now, consider the system in which the rewards of class $j$ are unbounded, i.e., $\bar{\rho}_j = \infty$, or in other words, for all $M > 0$, $P\{\rho_j \geq M\} > 0$. Then, we easily conclude that there are always preferred jobs of this class: We have interpreted the difference $D_n(0j)(x)$ as the burden of an additional class-$j$ job, and seen that a job of class $j$ bringing a reward of $\rho_j$ is admitted to the system if $D_n(0j)(x) \leq \rho_j$. Since the set $\mathcal{S}$ and $u(x)$ are both finite, $D_{max}(0j) = \max_{x\in\mathcal{S}}\{D_n(0j)(x)\} < \infty$. Then, $P\{\rho_j \geq D_{max}(0j)\} > 0$ so that there are always class-$j$ jobs who are willing to pay high enough prices to be served immediately whenever there is an available server. We have thus established the following:

**Proposition 1** *If the reward $\rho_j$ is unbounded, then there are preferred class-$j$ jobs.*

In the remainder of this section we assume that rewards of jobs from both classes are bounded, i.e., there exists $\bar{\rho}_j < \infty$, such that $P\{\rho_j > \bar{\rho}_j\} = 0$ and $P\{\rho_j \leq \bar{\rho}_j\} = 1$ for $j = 1, 2$. This is the more interesting case, both from the analysis and the application perspective.

The next theorem identifies sufficient conditions for the existence of class-2 preferred jobs.

**Theorem 2** *If*

$$\frac{\bar{\rho}_2}{\bar{\rho}_1} \geq \frac{\lambda_1}{\lambda_1 + \mu_2 + \beta},$$

*then $D_n(02)(x) \leq \bar{\rho}_2$ for all $x \in \mathcal{S}$ and for all $n$, hence there are preferred class-2 jobs.*

**Proof.** The proof is by induction. The initial condition $u_0(x) = 0$ for all $x \in \mathcal{S}$ clearly satisfies the statement. Assume that the statement is also true for period $n$, and consider period $n+1$. Now we use a sample path argument: Let system A be in state $x$ and system B in $x + e_2$ in period $n+1$. System A takes the optimal actions and system B rejects all jobs in period $n+1$. Consider an arrival. If system A also rejects either of the two classes, both systems remain in their current states, preserving the extra class-2 job. Acceptance of a class-1 job with a reward of $\rho_1$ to system A leads two systems to two different states $x + e_1$ and $x + e_2$ with a difference of $\rho_1$ in the value functions. If a class-2 job bringing a reward of $\rho_2$ is admitted to system A, then the two systems couple with a difference of $\rho_2$ in reward. With the departure of the additional class-2 job in system B, the systems again enter the same state, but with no difference in reward, whereas all other service completions keep the extra class-2 job in system A. Then:

$$\begin{aligned}
D_{n+1}(02)(x) &= u_{n+1}(x) - u_{n+1}(x + e_2) \\
&\leq \lambda_1 \max\{D_n(12)(x) + \rho_1, D_n(02)(x)\} + \lambda_2 \max\{\rho_2, D_n(02)(x)\} + \mu_2 \times 0 \\
&\quad + (c-1)\mu_2 \max_{y \in \mathcal{S}}\{D_n(02)(y)\} \\
&\leq \lambda_1 \max\{\bar{\rho}_1, \bar{\rho}_2\} + \lambda_2 \bar{\rho}_2 + (1 - \lambda_1 - \lambda_2 - \mu_2 - \beta)\bar{\rho}_2 \\
&\leq \lambda_1 \max\{\bar{\rho}_1 - \bar{\rho}_2, 0\} + (1 - \mu_2 - \beta)\bar{\rho}_2
\end{aligned}$$

where the first inequality is due to the coupling, the second inequality follows from the definition of $\bar{\rho}_j$, the induction hypothesis, part 2 of Lemma 2 and uniformization. If

$\bar\rho_2 \geq \bar\rho_1$, then the statement is proven. Otherwise, we have:

$$D_{n+1}(02)(x) \leq \lambda_1(\bar\rho_1 - \bar\rho_2) + (1 - \mu_2 - \beta)\bar\rho_2 = \bar\rho_2 - (\lambda_1 + \mu_2 + \beta)\bar\rho_2 + \lambda_1\bar\rho_1 \leq \bar\rho_2$$

where the last inequality is due to the assumption of the theorem. □

We next derive a similar sufficient condition for existence of class–1 preferred jobs. However, this requires some more work, since we have to consider an upper bound on $D_n(21)(x)$ simultaneously with the minimum offer for class 1, $D_n(01)(x)$.

**Lemma 4** *If*

$$\frac{\bar\rho_2(\mu_2 + \beta)}{\bar\rho_1(\mu_1 + \beta)} \leq \frac{\lambda_2 + \mu_2 + \beta}{\lambda_2},$$

*then for all $x \in \mathcal{S}$ and for all $n$:*

(1) $D_n(01)(x) \leq \bar\rho_1$.

(2) $D_n(21)(x) \leq \dfrac{\mu_2 - \mu_1}{\mu_2 + \beta}\bar\rho_1$.

**Proof.** We use induction on the number of transitions, $n$. Both statements are satisfied for $u_0(x) = 0$ for all $x \in \mathcal{S}$. Assume that both are true for $n$. Now we have to consider two pairs of systems, one for $D_{n+1}(01)(x)$ and the other for $D_{n+1}(12)(x)$.

(1) Consider the first pair: Assume that system A is in state $x$ and system B is in $x + e_1$ in period $n+1$, and we couple the two systems in such a way that system A follows the optimal policy, whereas system B rejects all jobs in period $n+1$. If upon an arrival system A also rejects either of the two classes, both systems remain in their current states, preserving the extra class-1 job. Acceptance of a class-1 job with a reward of $\rho_1$ to system A leads both systems to enter the same state with a difference of $\rho_1$ in reward. If a class-2 job bringing $\rho_2$ is admitted to system A, then the systems move to two different states $x + e_2$ and $x + e_1$ with a difference of $\rho_2$. With the departure of the additional class-1 job in system B, the systems again enter the same state but with no return, whereas all other service completions keep the difference between the two systems the same. Then:

$$\begin{aligned} D_{n+1}(01)(x) \quad &\leq \quad \lambda_1 \max\{\rho_1, D_n(01)(x)\} + \lambda_2 \max\{D_n(21)(x) + \rho_2, D_n(01)(x)\} \\ &\quad +(c\mu_2 - \mu_1) \max_{y \in \mathcal{S}}\{D_n(01)(y)\} \end{aligned}$$

$$\begin{aligned}
&\leq \lambda_1 \max\{\bar\rho_1, D_n(01)(x)\} + \lambda_2 \max\{D_n(21)(x) + \bar\rho_2, \bar\rho_1\} \\
&\quad +(1-\lambda_1-\lambda_2-\mu_1-\beta)\bar\rho_1 \\
&\leq \lambda_1\bar\rho_1 + \lambda_2 \max\left\{\frac{\mu_2-\mu_1}{\mu_2+\beta}\bar\rho_1 + \bar\rho_2, \bar\rho_1\right\} + (1-\lambda_1-\lambda_2-\mu_1-\beta)\bar\rho_1 \\
&\leq \lambda_2 \max\left\{\bar\rho_2 - \frac{\mu_1+\beta}{\mu_2+\beta}\bar\rho_1, 0\right\} + (1-\mu_1-\beta)\bar\rho_1
\end{aligned}$$

where the first inequality is due to coupling, the second due to the definition of $\bar\rho_j$, the induction hypothesis for $D_n(10)(x)$ and uniformization, and the third one follows from the induction hypotheses for $D_n(10)(x)$ and $D_n(21)(x)$. If $\bar\rho_2 \leq \frac{\mu_1+\beta}{\mu_2+\beta}\bar\rho_1$, the statement is proven; otherwise:

$$\begin{aligned}
D_{n+1}(01)(x) &\leq \lambda_2\bar\rho_2 - \lambda_2\frac{\mu_1+\beta}{\mu_2+\beta}\bar\rho_1 + (1-\mu_1-\beta)\bar\rho_1 \\
&\leq \bar\rho_1 + \lambda_2\bar\rho_2 - \frac{\mu_1+\beta}{\mu_2+\beta}\bar\rho_1(\lambda_2+\mu_2+\beta) \leq \bar\rho_1
\end{aligned}$$

where the last inequality is due to the assumption of the theorem. Thus, the first statement is true for all $x \in \mathcal{S}$ and for all $n \geq 0$.

(2) Now consider the second pair of systems: Let system A$'$ be in state $x+e_2$ and system B$'$ in $x+e_1$ in period $n+1$. System A$'$ takes the optimal actions and system B$'$ imitates all the actions of system A$'$ in this period. We, as in Lemma 2, couple the additional class-2 job, say job $d_2$, in system A$'$ with the additional class-1 job, say job $d_1$ in system B$'$, as well as all other service and interarrival times. Then, if $d_1$ leaves the system, which happens with probability $\mu_1$, $d_2$ also leaves. The departure of $d_1$ leads the system to couple with no reward, the departure of $d_2$ alone, which happens with probability $\mu_2-\mu_1$, takes the systems to two different states, $x$ and $x+e_1$ with no reward and whenever there is any other transition, both systems continue to have their additional jobs so that the difference between the two systems is due to changing a class-1 job to class 2:

$$\begin{aligned}
D_{n+1}(21)(x) &\leq \mu_1 \times 0 + (\mu_2-\mu_1)D_n(01)(x) + (\lambda_1+\lambda_2+(c-1)\mu_2)\max_{y\in\mathcal{S}}\{D_n(21)(y)\} \\
&\leq (\mu_2-\mu_1)\bar\rho_1 + (1-\mu_2-\beta)\frac{\mu_2-\mu_1}{\mu_2+\beta}\bar\rho_1 = \frac{\mu_2-\mu_1}{\mu_2+\beta}\bar\rho_1
\end{aligned}$$

where the first inequality is due to the coupling and the second follows by uniformization and the induction hypotheses for both $D_n(01)(x)$ and $D_n(21)(x)$. This proves the second part of the lemma. □

This lemma immediately leads to the following theorem which gives the sufficient conditions for class-1 jobs to be preferred:

**Theorem 3** *If*

$$\frac{\bar{\rho}_2(\mu_2+\beta)}{\bar{\rho}_1(\mu_1+\beta)} \leq \frac{\lambda_2+\mu_2+\beta}{\lambda_2},$$

*then there are preferred jobs of class 1.*

Theorem 2 and 3 present sufficient conditions to have preferred jobs of class 2 and 1, respectively. Theorem 2 implies that if the upper bound of both classes are the same, then there are always preferred class-2 jobs. This is very intuitive, since we would prefer to serve the faster jobs if both classes offer the same reward; i.e., in this case whenever a class-2 job offers a random reward of $\bar{\rho}_2$, (s)he will certainly be accepted if there is an idle server. Theorem 3 is not as intuitive, since the quantity $\frac{(\lambda_2+\mu_2+\beta)(\mu_1+\beta)}{\lambda_2(\mu_2+\beta)}$ is not necessarily greater than 1. However, we can still conclude that, in order to have preferred class-1 jobs, we need stronger conditions on the ratio of upper bounds, $\frac{\bar{\rho}_2}{\bar{\rho}_1}$, to accomodate the slowness of class-1 jobs.

# 5 Extensions

It is intuitively expected that the required minimal offer is increasing with the congestion level in the system. Thus, it would be interesting to show that the minimal offer for class $j$ is increasing in the number of class-$j$ jobs as well as in the number of class-$k$ jobs; the latter is equivalent to concavity of $u_n$ in $x_j$ for fixed $x_k$, $k \neq j$. We also note that concavity of $u_n$ would imply monotonicity of the threholds on the number of class-$j$ customers in the system when admission of a class-$k$ customer is considered, with $j \neq k$.

The results of this paper can be extended for a general arrival process, which can be modeled as an embedded MDP at arrival times. This allows a more realistic modeling of computer and communication systems.

We can also consider controlling the admission of jobs into the system via pricing. Thus, instead of rejecting the jobs, we can propose a price, which may or may not depend on the state of the system, for which we are willing to serve the incoming job. Each customer class has a probability distribution so that an incoming customer with a request for job

$j$ decides to receive service at a price of $\rho$ with probability $p_j(\rho)$, where this probability distribution is known to us.

The present system can be considered under batch arrivals, which are appropriate for computer and communication systems. Lippman & Ross (1971) are the first to consider the batch arrivals for a single server no waiting room system. Örmeci & Burnetas (1999) have also considered a system under batch arrivals which has $c$ identical parallel servers and two classes of jobs with fixed rewards and rejection costs. They provide partial characterizations of the optimal policy and establish certain monotonicity properties. Similar results may be obtained with random rewards.

## References

Carrizosa, E., Conde, E. & Munoz-Marquez, M. (1998) , 'Admission policies in loss queueing models with heterogeneous arrivals', *Management Science* **44**, 311–320.

Ghoneim, H. A. & Stidham, S. (1985) , 'Control of arrivals to two queues in series', *Euro. Jour. Oper. Res.* **21**, 399–409.

Keilson, J. (1970) , 'A simple algorithm for contract acceptance', *Opsearch* **7**, 157–166.

Ku, C. & Jordan, S. (1997) , 'Access control to two multiserver loss queues in series', *IEEE Transactions on Automatic Control* **42**, 1017–1023.

Lippman, S. A. (1975*a*) , 'Applying a new device in the optimization of exponential queueing systems', *Operations Research* **23**, 687–710.

Lippman, S. A. (1975*b*) , 'On dynamic programming with unbounded rewards', *Management Science* **21**, 1225–1233.

Lippman, S. A. & Ross, S. M. (1971) , 'The streetwalker's dilemma: A job shop model', *SIAM J. Appl. Math.* **20**, 336–342.

Lippman, S. A. & Stidham, S. M. (1977) , 'Individual versus social optimization in exponential congestion system', *Operations Research* **25**, 233–247.

Miller, B. (1971) , 'A queuing reward system with several customer classes', *Management Science* **16**, 234–245.

Miller, B. L. & Buckman, A. G. (1987) , 'Cost allocation and opportunity costs', *Management Science* **33**, 626–639.

Naor, P. (1969) , 'On the regulation of queue size by levying tolls', *Econometrica* **37**, 15–24.

Örmeci, E. L. (1998) , *Idling Rules for Queues with Preferred Customers*, Ph.D. thesis, Case Western Reserve University, Cleveland.

Örmeci, E. L. & Burnetas, A. (1999) , *Admission Policies for a Two Class Loss System with Batch Arrivals*, EURANDOM Technical Report 99-052, Eindhoven.

Örmeci, E. L., Burnetas, A. & van der Wal, J. (1999) , *Admission Policies for a Two Class Loss System*, EURANDOM Technical Report 99-050, Eindhoven.

Ross, K. W. (1995) , *Multiservice Loss Models for Broadband Telecommunication Networks*, Springer-Verlag, Great Britain.

Walrand, J. (1988) , *Introduction to Queueing Networks*, Prentice Hall, Englewood Cliffs, N.J.

Xu, S. H. & Shantikumar, J. G. (1993) , 'Optimal expulsion control-a dual approach to admission control of an ordered-entry system', *Operations Research* **41**, 1137–1152.