Preferences, Risk Neutrality, and Utility Functions

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Abstract

A binary preference relation $\succeq$ on a real vector space V with zero element $\mathbf{0}$ satisfying four (natural) axioms induces a utility function $U = f \circ u : V \to R$ where $u \colon V \to R$ is linear as a map of vector spaces and $f : R \to R$ is weakly monotone. The weak ordering and non-triviality axioms are familiar, but the continuity axiom does *not* require topological assumptions. The fourth axiom (decomposition) is $X - Y \succeq (\preceq)\ \mathbf{0} \Rightarrow X \succeq (\preceq)\ Y$ $(X, Y \in V)$. The utility function $U : R \to R$ can be taken to be linear if and only if $\succeq$ also satisfies the converse of the fourth axiom. When V is a real vector space of stochastic processes and $\mathbf{0}$ is the zero process, it is known that the four axioms imply the existence of discount factors and, with the converse, the linearity of an intra-period utility function. Thus preferences correspond to discounting and are *not* risk neutral only if the converse of the fourth axiom is *not* satisfied. One consequence is that, within the developed context, a utility function that is not risk neutral can be replaced by one that is risk neutral and still be consistent with the underlying preference relation. Previous axiomatizations of the discounted utility model by T. C. Koopmans and others unwittingly imply risk neutrality because the axiom sets include the four and the converse.

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1 Introduction

We concern ourselves with preference orderings on sets, with particular attention to the presence or absence of risk neutrality, and the resulting representation of utility functions. We suppose a weak ordering on a real vector space with minimal basic properties. If the weak ordering can be represented by a utility function U, various definitions of risk neutrality are equivalent to the affinity of U (Keeney and Raiffa, 1976). However, we follow Miyamoto and Wakker (1996) in emphasizing the weak ordering as the primary object rather than the utility function. This approach elucidates a number of points. We develop results in a context where there are many (an infinite number) of utility functions corresponding to any ordering, some of which may be risk neutral and some not. This is another justification for the widely held belief that preference relations are a more basic concept than utility functions. Utility functions that are not risk neutral have been used since Daniel Bernoulli's treatment of the St. Petersburg paradox; for examples see Markowitz (1959); Rubinstein (1976); Pliskin et al. (1980); Garber and Phelps (1997); Krysiak and Krysiak (2006).

In our context a utility function that is not risk neutral can be replaced by one that is risk neutral and yet be consistent with the same preference relation. A simple consequence of our treatment is to completely clarify the situation, and characterize those preference relations for which there is a risk-neutral utility function. Roughly speaking, a preference relation (i.e., a weak ordering) has a risk-neutral utility function if and only if it does not have any "indifference regions." We note that we make no topological assumptions on the space of preferences, yet we obtain sufficient conditions for the existence of a utility function. Also, our treatment leads to conditions for *preferences over time and under risk to correspond to discounting without risk neutrality.*

The authors have benefited from the comments and suggestions of a number of interested individuals towards, in part, these designs. We particularly acknowledge Vera Tilson. The present paper is the result of revisions based on such feedback. We note that similar results could be derived, *mutatis mutandi*, for mixture spaces (Hausner, 1954; Herstein and Milnor, 1953). Although our treatment is abstract, the application to the discounted utility model has particular interest. Deterministic and stochastic versions of this model, as developed by Samuelson and successors, are discussed in more detail below. In particular, we ask (and answer): *Under what conditions in this model do preferences over time and under risk correspond to discounting* without *risk neutrality?*

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Below, we set the abstract framework and state the key result, and in the following section, explicate the concrete implementation for discounted utility. The proof of the key result follows a discussion section.

2 Axioms, definitions, and main result

Our main purpose is to elicit the structure of a utility function when a binary relation $\succeq$ on a real vector space V with zero $\mathbf{0}$ satisfies the following axioms:

(A1) *rationality*: $\succeq$ is a weak ordering (preorder; reflexive, transitive, complete) on V; $\succ$ denotes the associated strong ordering and $\sim$ the associated equivalence,
(A2) *decomposition*: $X - Y \succeq (\preceq)\ \mathbf{0}$ implies $X \succeq (\preceq)\ Y$,
(A3) *continuity*: for any X, $Y \in V$, the sets $\{\alpha \in R : \alpha X - Y \succeq (\preceq)\ \mathbf{0}\}$ are closed,
(A4) *non-triviality*: there exists an element $X_0 \in V$ such that $X_0 \succ \mathbf{0}$.

The presence or absence of the converse of decomposition:

(A2^c) $X \succeq (\preceq)\ Y$ implies $X - Y \succeq (\preceq)\ \mathbf{0}$,

is a key concern.

A *pseudo-utility function* is a function $u : V \to R$ such that $u(X) \geq u(Y)$ implies $X \succeq Y$, or contrapositively, $X \succ Y$ implies $u(X) > u(Y)$. (This definition differs slightly from those in Subiza and Peris (1998), Peleg (1970), and Candeal et al. (1998).) A pseudo-utility function can be overly discriminating; that is, it may be that $X \sim Y$, but $u(X) \neq u(Y)$. A *utility function* is a pseudo-utility function $U : V \to R$ such that $X \succeq Y$ if and only if $U(X) \geq U(Y)$.

Theorem: *For every ordering satisfying (A1)–(A4), there exists a utility function of the form $U = f \circ u : V \to R$ in which $u: V \to R$ is a linear pseudo-utility function. Also, $f : R \to R$ is weakly monotonic and can be taken to be linear if and only if (A2^c) holds.*

There is a partial converse, discussed below. The immediate decision-theoretic consequence of the theorem is that preferences satisfying the four axioms are risk neutral if and *only if* the converse (A2^c) of decomposition is satisfied too. Only sufficiency in a restricted setting had previously been established (Sobel, 2006).

3 Discounting

A major motivation and application is the theory of discounted utility where the theorem has implications for discounting and risk neutrality. Let I denote the natural numbers, $T \in I$, $I_T = \{1, 2, .., T\}$, and V be the set of stochastic sequences $X = (X_1, X_2, ..., X_T)$ defined on a probability space (Ω, F, P) with $X_t(\omega) \in R^M$ for all $(t, \omega) \in I_T \times \Omega$. For $X \in V$, $Y \in V$, and $b \in R$, define $X + Y \in V$ and $bX \in V$ with component-wise addition and multiplication, respectively. Let θ be the zero vector in R^M, so V is a real vector space with zero element $\mathbf{0} = (\theta, \theta, ...) \in V$. Let S be the set of random vectors with sample space R^M, and for $C \in S$ denote $(C, \theta, \theta, ...)$ as $(C, \mathbf{0})$. Then $(V, \succeq)$ induces a preference relation $\gg$ on S: $A \gg B$ if and only if $(A, \mathbf{0}) \succeq (B, \mathbf{0})$. Let e_m be the m^{th} unit vector in R^M and $e_{mt} \in V$ be the perturbation of $\mathbf{0}$ where e_m replaces the t^{th} θ. The *discounted utility model* is

$$X \succeq Y \Leftrightarrow E[\Sigma_t \beta_t g(X_t)] \geq E[\Sigma_t \beta_t g(Y_t)] \tag{1}$$

where $X_t = (X_{1t}, ..., X_{Mt}) \in S$, E denotes expected value, $g : R^M \to R$ is an *intra-period* utility function, and the $\beta_t \in R$ are discount factors. An *intra-period utility function* satisfies the definition of a utility function except that $(\gg, S)$ replaces $(\succeq, V)$. This model, the basis of many economics applications, was suggested by Samuelson (1937) in a deterministic setting where it was axiomatized by Koopmans (Koopmans, 1960, 1972; Koopmans et al., 1964). In the stochastic case, the model emerges from results concerning multiattribute preference orderings and utility functions that are unified and generalized by the axiomatization in Miyamoto and Wakker (1996). These axiomatizations unwittingly imply that f is affine, i.e. preferences are risk neutral.

Sufficient conditions for discounting and the existence of an intra-period utility function imply risk neutrality (Sobel, 2006). That is, there are two consequences if $(V, \succeq)$ satisfies axioms (A1)–(A3) and $e_{mt} \succ \mathbf{0}$ for all $t \in I_T$ and m (which implies (A4)). Unlike the scalar discount factors above, for $\alpha_t = (\alpha_{1t}, \cdots, \alpha_{Mt}) \in R^M$ let $\alpha_t X_t$ denote $(\alpha_{1t} X_{1t}, \cdots, \alpha_{Mt} X_{Mt})$. Firstly, for each t there exists a unique $\alpha_t \in R^M$ with positive components such that $X \succeq Y$ if and only if $\Sigma_t \alpha_t X_t \gg \Sigma_t \alpha_t Y_t$. Secondly, the following properties are equivalent:

- $(A2^c)$,
- existence of an intra-period utility function, and
- risk neutrality.

Risk neutrality here means the existence of $\gamma \in R^M$ such that $X \succeq Y$ if and only if

$$\gamma \cdot E(\Sigma_t \beta_t X_t) \geq \gamma \cdot E(\Sigma_t \beta_t Y_t) \tag{2}$$

for all X and Y for which the expectations exist finitely. Further assumptions yield the familiar geometric form $\beta_{mt} = (\beta_m)^t$ and $\beta_m < 1$. In contrast to the restricted setting in Sobel (2006), the theorem shows that (A2^c) is fundamentally equivalent to risk neutrality, and that the equivalence remains valid if I_T is replaced by I. The question asked in §1 was: *Under what conditions do preferences over time and under risk correspond to discounting* without *risk neutrality?* Our answer is: axioms (A1)–(A4) without (A2^c).

The theorem might seem to imply that if preferences satisfy (A1)–(A4) without (A2^c), then there is a nonlinear function $W : R^M \to R$ such that

$$X \succeq Y \Leftrightarrow E[W(\Sigma_t \beta_t X_t)] \geq E[W(\Sigma_t \beta_t Y_t)] \tag{3}$$

However, (observation of James E. Smith) the theorem implies that there would exist an intra-period utility function if (3) were valid. So preferences would be risk neutral implying (2).

The comparison in (1) is additively separable, so the separability literature (Blackorby et al., 1998) is relevant but does not address the structure of the separable terms. Extensive and largely unrelated literatures either take the intra-period utility function as given and discuss the choice of a discount factor (Arrow et al., 1995; Lind et al., 1982; Portney and Weyant, 1999), or investigate the existence and properties of the intra-period utility function (Mehta, 1998). The relationship of the issues in this paper to the intertemporal resolution of uncertainty (Johnson and Donaldson, 1985; Kreps and Porteus, 1979; Machina, 1989) is unclear.

4 Discussion

Here we note some aspects of the axioms, and discuss examples, applications, and variations.

4.1 Topology

Some natural applications involve infinite-dimensional V, so we make no dimensional assumption. Past axiomatic constructions of utility functions on uncountable outcome sets have followed two routes (Fishburn, 1979). Some endow $(V, \succeq)$ with a topology and posit a continuity axiom (Bridges and Mehta, 1995). Another approach, if the dimension of V is at least two, is algebraic and is based on the assumption that outcomes that differ in some dimensions can be offset with compensating differences in other dimensions (Luce and Tukey, 1964). We make no assumption regarding compensating differences and the continuity axiom is novel because we make no topological assumptions. Note that (A3) refers only to the topology of the real numbers, and V is not endowed with a topology. Of course, any finite-dimensional vector space over R carries a unique natural (product) topology, but this topology is not pertinent to our discussion.

4.2 Utility functions

At least two definitions of a utility function are current in the literature. Let S and T be binary relations on sets V and B, respectively. A function $g : V \to B$ is an *order homomorphism* if $XSY \Rightarrow g(X)Tg(Y)$ for all $X, Y \in V$, and it is an *order isomorphism* if $XSY \Leftrightarrow g(X)Tg(Y)$ for all $X, Y \in V$. When $B = R$, a utility function has recently been defined as an order homomorphism (e.g., Bridges and Mehta (1995, page 5), Mehta (1998), and Vind (2003)) and as an order isomorphism (e.g., Bridges and Mehta (1995, page 27), Vind (2003), and Ok (2007)). The extant existence proofs with both definitions use topological properties of V (cf. references cited in this paragraph). Henceforth, we say that *preferences are risk neutral* if there is a utility function $U = f \circ u : V \to R$ in which $u \colon V \to R$ is a linear pseudo-utility function and $f : R \to R$ is linear.

4.3 Partial orderings

A real vector space with a binary relation is said to be *partially ordered* if it has a cone property that $x \succeq y$ implies $\alpha x \succeq \alpha y$ for all $\alpha \geq 0$, and it satisfies antisymmetry, (A1), (A2), and (A2^c). There exists a linear pseudo-utility function $u \colon V \to R$ if a vector space is partially ordered and V has additional properties (Hausner, 1954; Hausner and Wendel, 1952). However, we do not assume that V is partially ordered and the effect of the absence or not of (A2^c) is a major point of interest. The theorem yields the existence of a linear pseudo-utility function without requiring (A2^c). It follows from part 4 of the lemma in the proof that the cone property is redundant in the definition if a partially ordered vector space (Hausner, 1954; Hausner and Wendel, 1952; Peressini, 1967; Aliprantis et al., 1989) satisfies (A3).

4.4 Examples

Note that a non-trivial linear function $u : V \to R$ defines an order $\succeq_u$, via $X \succeq_u Y$ if $u(X) \geq u(Y)$. This order satisfies (A1)–(A4) and also (A2^c). On the other hand, consider the following examples.

- Let $V = R$. Define $X \succeq Y$ if $X \geq Y$ or if $X \geq 1$. This order satisfies

(A1)–(A4), but not (A2^c). The function $U(X) = \min\{X, 1\}$ is a utility function.

- This example is the same as above, except also $1 \prec X$ for all $X > 1$. The utility function has a jump at 1:

$$U(X) = \begin{cases} X, & X \leq 1, \\ 2, & X > 1. \end{cases}$$

We see below that these examples are prototypical. In particular, how an order fails (A2^c) is made apparent. In fact, the second example is somewhat exotic and does not occur if (A3) is strengthened slightly as follows:

(A3') the sets $\{\alpha \in R : \alpha X - Y \succeq (\preceq)\ Z\}$ are closed for any X, Y, $Z \in V$.

The second example above does not satisfy (A3') with $Y = 0$, $X = 1$, $Z = 2$; the set in (A3') is the open half-line $(1, \infty)$.

4.5 Variants

D. Turcic (unpublished) has suggested replacing (A2) and/or (A2^c) with

(A$\tilde{2}$) *monotonicity*: For all X, $Y \in V$ such that $X - Y \succeq 0$ and α, $\beta \in [0, 1]$, $\alpha \geq \beta \Leftrightarrow$

$$\alpha X + (1 - \alpha)Y \succeq \beta X + (1 - \beta)Y \tag{4}$$

We note that (A$\tilde{2}$) is stronger than the combination of (A2) with ($A2^c$) (let $\alpha = 1$, $\beta = 0$). Indeed, (A$\tilde{2}$) is strictly stronger than the combination; the first example above satisfies (A2) but not (A$\tilde{2}$) (let $X = 2$, $Y = 0$, $\alpha = .9 < 1 = \beta$). On the other hand, (A1)–(A4) and (A2^c) imply (A$\tilde{2}$). For by our theorem, (A1)–(A4) and (A2^c) imply there is a *linear* utility function $U : V \to R$, in which case (4) is equivalent to $(\alpha - \beta)U(X - Y) \geq 0$. We note that the following variant of (A$\tilde{2}$) implies (A2^c):

(A$\tilde{\tilde{2}}$) For all X, $Y \in V$ such that $X - Y \succ 0$ and α, $\beta \in [0, 1]$, $\alpha > \beta$ implies

$$\alpha X + (1 - \alpha)Y \succ \beta X + (1 - \beta)Y$$

For note that (A2^c) is equivalent to its contrapositive: $X - Y \succ 0 \Rightarrow X \succ Y$, and again, let $\alpha = 1$, $\beta = 0$.

5 Proof of main result

For use below, we establish some elementary technical properties of the ordering.

Lemma. *Axioms (A1), (A2), and (A3) imply the following:*

(1) $W \succeq (\preceq, \sim)\ \mathbf{0}$ *implies* $W + Z \succeq (\preceq, \sim)\ Z$ *for all* $Z \in V$,
(2) *if* $X \succeq (\preceq, \sim)\ \mathbf{0}$ (resp. $X \succ (\prec)\ \mathbf{0}$) *and* $Y \succeq (\preceq, \sim)\ \mathbf{0}$, *then* $X + Y \succeq (\preceq, \sim)\ \mathbf{0}$ $(X + Y \succ (\prec)\ \mathbf{0})$,
(3) *if* $X \succeq (\preceq, \sim)\ \mathbf{0}$ (resp. $X \succ (\prec)\ \mathbf{0}$), *then* $-X \preceq (\succeq, \sim)\ \mathbf{0}$ $(-X \prec (\succ)\ \mathbf{0})$,
(4) *if* $X \succeq (\preceq, \sim)\ \mathbf{0}$, *then* $bX \succeq (\preceq, \sim)\ \mathbf{0}$ *for all real* $b > 0$.

Proof. For part 1: use axiom (A2) with $X = W + Z$ and $Y = Z$. For $\sim$, combine the cases $\succeq$ and $\preceq$. For part 2: part 1 implies $X + Y \succeq X \succeq (\succ)\ \mathbf{0}$, and similarly for the opposite order. The third follows immediately. For part 4: use induction on part 2 to see that $nX \succeq (\preceq)\ \mathbf{0}$ for all non-negative integers n. Thus $(n/m)X \succeq (\preceq)\ \mathbf{0}$ for all non-negative rationals n/m. Finally, use (A3) (with $Y = \mathbf{0}$) to establish that $bX \succeq (\preceq)\ \mathbf{0}$ for all non-negative b. This proves the lemma.

Proof of theorem. Given an order satisfying (A1)–(A4), we first construct a linear pseudo-utility function u. We claim that for any Y, there exists a unique real c such that $cX_0 - Y \sim \mathbf{0}$. First note that there is at most one such c for any Y. For if $cX_0 - Y \sim \mathbf{0}$, $c'X_0 - Y \sim \mathbf{0}$ then (Lemma, parts 3 and 2), $(c - c')X_0 \sim \mathbf{0}$, and (Lemma, part 4), $c - c' = 0$.

Next we establish the existence of c. Any $Y \in V$ satisfies exactly one of $Y \succ \mathbf{0}$, $Y \sim \mathbf{0}$ or $Y \prec \mathbf{0}$. If $Y \sim \mathbf{0}$, set $c = 0$. If $Y \succ \mathbf{0}$, consider the set $A = \{\alpha : \alpha X_0 - Y \succ \mathbf{0}\}$. We claim $A \neq \emptyset$. For if $X_0 - (1/\alpha)Y \preceq \mathbf{0}$ for all large α, then by Lemma, part 3 and (A3), $X_0 \preceq \mathbf{0}$, which contradicts (A4). Moreover, if $\alpha < 0$, (Lemma, parts 4, 3 and 2) $\alpha \notin A$. Thus let $c = \inf\{\alpha \in A\} \geq 0$. By (A3) $cX_0 - Y \succeq \mathbf{0}$. If $\alpha < c$, then $\alpha X_0 - Y \preceq \mathbf{0}$, so (A3 again) $cX_0 - Y \preceq \mathbf{0}$. Thus $cX_0 - Y \sim \mathbf{0}$. If $Y \prec \mathbf{0}$, use the previous argument on $-Y$.

We can thus define $u(Y) = c$. We next establish that u is linear. If $cX_0 - Y \sim \mathbf{0}$, then (Lemma, part 4) $bcX_0 - bY \sim 0$ for all real b, so that $u(bY) = bu(Y)$. Also, if $cX_0 - Y \sim \mathbf{0}$ and $c'X_0 - Y' \sim \mathbf{0}$, then (Lemma, part 2) $(c + c')X_0 - (Y + Y') \sim \mathbf{0}$, so that $u(Y + Y') = u(Y) + u(Y')$. Hence u is linear. We also establish that u is a pseudo-utility. If $Y \succ Y'$, then (contrapositive of A2), $\mathbf{0} \prec Y - Y' = (cX_0 - Y') - (cX_0 - Y)$. Since $cX_0 - Y \sim \mathbf{0}$, (Lemma, part 2) $cX_0 - Y' \succ \mathbf{0}$, so $u(Y') < c$. Thus u is a pseudo-utility.

Finally, we construct a utility function U. This amounts to showing that a certain quotient of R is again a copy of R. This seems like it should be a standard known result; however, we could not locate a reference, and include the construction for compleness. Also, we give an explicit construction, which could be useful in explicit applications.

Define an equivalence relation on R by $x \equiv y$ if there are $X \in V$ and $Y \in V$ with $u(X) = x$, $u(Y) = y$ and $X \sim Y$. Suppose $X \sim Y$ with $u(X) \geq u(Z) \geq u(Y)$. Then $X \succeq Z \succeq Y \sim X$, so that (A1) $X \sim Z \sim Y$. That is, all elements with u values between those of X and Y, inclusive, are equivalent under the order. Thus the relation $\equiv$ is well-defined (does not depend on the choices of X and Y), and moreover, for any x, the set of y equivalent to x is a single point or an interval $I = I_x$. The idea is to define a monotonic function $f : R \to R$ such that $f(x) = f(y)$ if and only if $x \equiv y$. Then $U = f \circ u : V \to R$ is the desired utility function.

For numbers $x \in R$, denote $x \succ y$, etc., if for any X, Y with $u(X) = x$, $u(Y) = y$, $X \succ Y$, etc. By the above, this ordering is well-defined. For any interval I, let x_I^ℓ (for left) and x_I^r (for right) denote the end points of I. Call an end point $x_I^{\ell,r}$ of *regular* if $x_I^{\ell,r} \sim x$ for $x \in I^\circ$ (the interior of I), and *irregular* otherwise (in the examples above, $x = 1$ is regular in the first case and irregular in the second). Note that a point can be the right boundary point of one interval and the left boundary point of another, and irregular with respect to either or both intervals.

We begin with the simplest case, and the only one really relevant for applications. Assume there are no irregular points and only finitely many intervals I on any bounded set of R, equivalently the lengths of the intervals I are bounded away from zero. A mathematical point is that the end points of the intervals I do not have any accumulation points. Define $g : R \to R$ by $g = 0$ on the union of the intervals I and $g = 1$ otherwise. The function g is the density of an absolutely continuous measure with characteristic function $f(x) = \int_0^x g(y)\,dy$, and $U = f \circ u$ is the desired utility function.

More generally, we define the function f in terms of Lebesgue integrals of non-negative measures on R. For g defined as above, let $\mu_1 = g\,dx$, where dx is the usual Lebesgue measure. Let $\delta(x)$ denote the delta function (as a measure) at x. Let $\mu_2 = \sum \ell(I)\delta_{x_I^{\ell,r}}$, where the sum is over all irregular end points of intervals I and $\ell(I)$ is the length of I.

Before defining μ_3, an example: Let $V = R$. Write any real number x between 0 and 1 in trinary notation $x = \sum_{i=1}^\infty a_i 3^{-i}$, where $a_i = 0$, 1 or 2. Suppose the I° are precisely the intervals of numbers with no '1' in the expansion. The I° are the middle-third intervals in the construction of the Cantor set, and the complement of $\cup I^\circ$ on the interval $[0, 1]$ is the standard Cantor set.

For convenience, we suppose all the intervals I are contained in a bounded set of R. For the full case, one performs the construction below on a sequence of bounded sets; the details are left to the reader. For any (small) $L > 0$, let $N(L) < \infty$ be the number of intervals with $\ell(I) \geq L$ (it is here we require the intervals to be in a bounded set). Define

$$\mu_3 = \lim_{L \to 0} 2^{-N(L)-1} \sum_{\ell(I) \geq L} \Big(\delta(x_I^\ell) + \delta(x_I^r) \Big).$$

The measure μ_3 is supported on the set of accumulation points of the end points of the intervals I. In the example above, μ_3 is the standard Cantor measure, and the Lebesgue integral $\int_0^x d\mu_3$ is the standard Cantor function on $[0, 1]$.

Define $\mu = \mu_1 + \mu_2 + \mu_3$, and let

$$f(x) = \frac{1}{2}\Big(\int_0^{x+} d\mu + \int_0^{x-} d\mu \Big) = \frac{1}{2}\Big(\lim_{y \downarrow x} \int_0^y d\mu + \lim_{y \uparrow x} \int_0^y d\mu \Big).$$

It is routine to establish that f has the desired property, namely $f(x) = f(y)$ if and only if $x \sim y$.

6 Notes

- If x is not an irregular endpoint of an interval, then f is the Lebesgue integral $f(x) = \int_0^x d\mu$. Thus if there are no irregular points, the function f is continuous. At each irregular point, the function f has a jump of size $\ell(I)/2$ on each side (thus the utility function of the proof is slightly different than in the example).
- If there are no irregular points, and the lengths $\ell(I)$ are bounded away from zero, the function f can be made smooth (C^∞), by redefining g. Details are left to the reader.
- The function f is defined as a Lebesgue integral, but can be defined as a Stieltjes integral. If $\mu_2 = \mu_3 = 0$, it can be defined as a Riemann integral.
- The pseudo-utility function u is unique up to scale. It is determined up to scale by the set $\{X : X \succeq \mathbf{0}\}$.
- For the two examples above, the pseudo-utility functions are both the identity from R to R.
- In the construction of U from u, the number 0 is not in any interval I_x, otherwise (A2) would be violated. Conversely, given u, any set of disjoint intervals I_x, none of which contains 0, determines, via the resulting U, an order satisfying (A1)–(A4).
- If (A2^c) obtains, then the equivalence relation $\equiv$ is trivial; $x \equiv y$ if and only if $x = y$. In this case, the utility function can be assumed linear.

Conversely, if the utility function is linear, the equivalence relation $\equiv$ is trivial, and (A2^c) obtains.

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